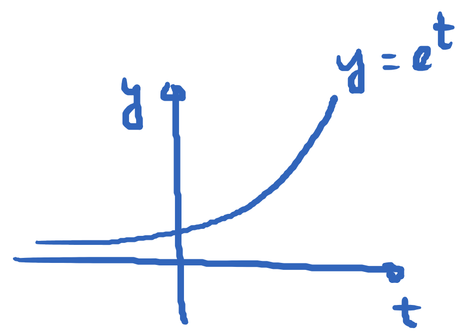
(2) $\displaystyle\int_{-\infty}^{0} xe^x\,dx = \lim_{t\to-\infty}\int_t^0 xe^x\,dx$

$\begin{cases} u = x \\ dv = e^x\,dx \end{cases} \qquad \begin{cases} du = dx \\ v = e^x \end{cases}$

$\displaystyle= \lim_{t\to-\infty}\left(xe^x\Big|_t^0 - \int_t^0 e^x\,dx \right)$

$\displaystyle= \lim_{t\to-\infty}\left(-te^t - \underbrace{e^0}_{1} + e^t\right)$

$\displaystyle= -1 + \lim_{t\to-\infty}\left(-te^t + e^t\right)$



$\displaystyle= -1 + \lim_{t\to-\infty} e^t\cdot(1-t).$

to use L'Hopital.

$\displaystyle= -1 + \lim_{t\to-\infty}\frac{1-t}{\frac{1}{e^t}} = -1 + \lim_{t\to-\infty}\frac{1-t}{e^{-t}}$

$$= -1 + \lim_{t\to-\infty} \frac{-1}{-e^{-t}} = -1$$

0

③ $$\int_{-\infty}^{\infty} x e^{-x^2}\,dx = \int_{-\infty}^{0} x e^{-x^2}\,dx + \int_{0}^{\infty} x e^{-x^2}\,dx$$

$-\frac{1}{2}$ $\quad \frac{1}{2}$

$$\int_{-\infty}^{0} x e^{-x^2}\,dx = \lim_{t\to-\infty} -\frac{1}{2}\int_{t}^{0} -2x\, e^{-x^2}\,dx$$

$u = -x^2$

$du = -2x\,dx$

$\int e^u\,du = e^u$

$$= \lim_{t\to-\infty} -\frac{1}{2}\cdot e^{-x^2}\Big|_t^0$$

$$= \lim_{t\to-\infty} -\frac{1}{2}\left[e^0 - e^{-t^2}\right]$$

$$= \lim_{t\to-\infty} -\frac{1}{2}\cdot\left[1 - \cancel{e^{-t^2}}\right]$$

0

$$= -\frac{1}{2}.$$

$$\int_0^\infty x e^{-x^2}\,dx = \lim_{t\to\infty} \int_0^t x e^{-x^2}\,dx$$

$$= \lim_{t\to\infty} \left(-\frac{1}{2} e^{-x^2}\right)\Big|_0^t$$

$$= \lim_{t\to\infty} \left(-\frac{1}{2}\cdot\left(\overset{\to 0}{e^{-t^2}} \underbrace{- e^{0}}_{-1}\right)\right)$$

$$= \frac{1}{2}$$

So, $\int_{-\infty}^{\infty} x e^{-x^2}\,dx = -\frac{1}{2} + \frac{1}{2} = 0$

Type 2 Improper Integrals:

Integrals of Discontinuous Functions.

E.g. $\displaystyle \int_{-2}^{3} \frac{1}{x^3}\,dx = \int_{-2}^{3} x^{-3}\,dx = \frac{x^{-2}}{-2}\Big|_{-2}^{3}$

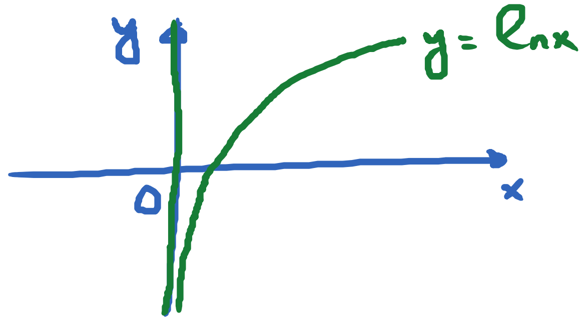
$\displaystyle = -\frac{1}{2x^2}\Big|_{-2}^{3} = -\frac{1}{18} + \frac{1}{8} = \cdots$

$\boxed{\text{Wrong.}}$

0

$\frac{1}{x^3}$ is discontinuous at 0, a point in $[-2, 3]$.

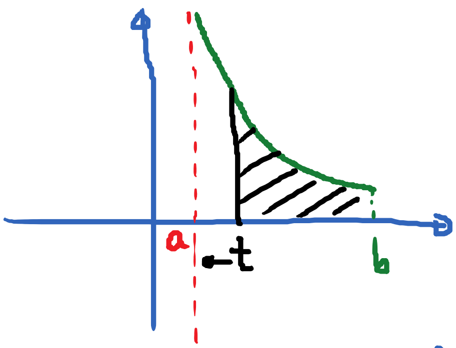
E.g. $\displaystyle \int_0^1 \ln x\,dx.$



E.g. $\displaystyle \int_0^3 \frac{dx}{\sqrt{3-x}}$ $\quad \frac{1}{\sqrt{3-x}}$ has a discontinuity at 3.

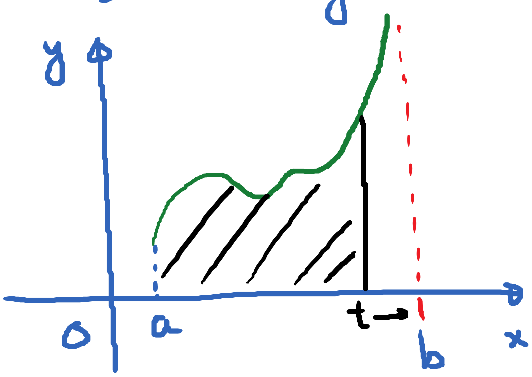
$\longrightarrow$ None of these are proper integrals.

* Discontinuity at left endpoint. $\int_a^b f(x)\,dx$, f has a discontinuity at $x=a$.



$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

* Discontinuity at the right endpoint.



$\int_a^b f(x)\,dx$, f has a discontinuity at $x=b$.

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

* Discontinuity at a point c in (a,b):

f has a discontinuity at c in (a,b)

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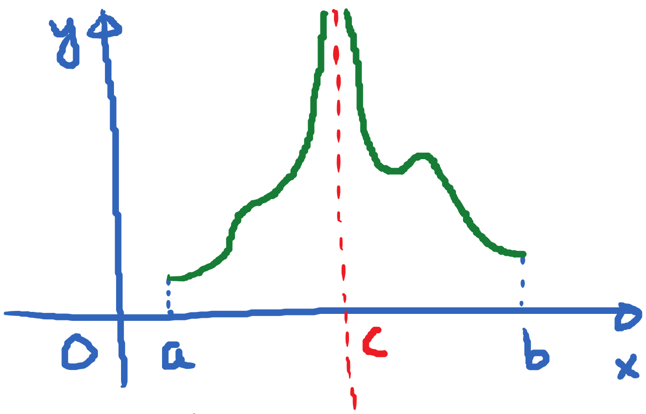

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \lim_{t \to c^-} \int_a^t f(x)\,dx + \lim_{t \to c^+} \int_t^b f(x)\,dx$$

$\int_a^b f(x)\,dx$ converges if both of these limits exist and finite.

E.g. $$\int_{-2}^{3} \frac{1}{x^3}\,dx = \int_{-2}^{0} \frac{1}{x^3}\,dx + \int_{0}^{3} \frac{1}{x^3}\,dx$$

$$\int_{-2}^{0} \frac{1}{x^3}\,dx = \lim_{t \to 0^-} \int_{-2}^{t} \frac{1}{x^3}\,dx = \lim_{t \to 0^-} \left(-\frac{1}{2x^2}\right)\Bigg|_{-2}^{t}$$

$$= \lim_{t \to 0^-} \left(-\frac{1}{2t^2} + \frac{1}{8}\right) = -\infty$$

So, $\int_{-2}^{3} \frac{1}{x^3}\,dx$ diverges.

Ex. $\displaystyle\int_0^3 \frac{dx}{\sqrt{3-x}}$. converges or diverges?

$$= \lim_{t\to 3^-} \int_0^t \frac{dx}{\sqrt{3-x}} = \lim_{t\to 3^-} -\int_0^t (3-x)^{-1/2}\,(-dx)$$

$$= \lim_{t\to 3^-} \left(-2\sqrt{3-x}\right)\Big|_0^t$$

$u = 3-x$

$du = -dx$

$$-\int u^{-1/2}\,du = -\frac{u^{1/2}}{1/2} = -2\sqrt{u}$$

$$= \lim_{t\to 3^-} \left(-2\sqrt{3-t} + 2\sqrt{3}\right)$$

$= 2\sqrt{3}$. So, $\displaystyle\int_0^3 \frac{dx}{\sqrt{3-x}}$ converges to $2\sqrt{3}$.