

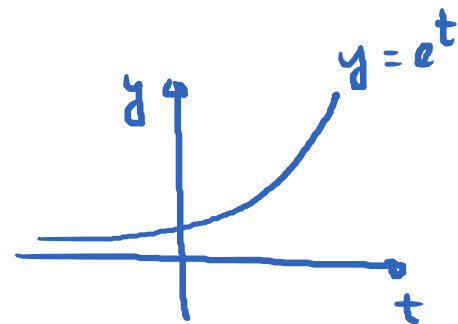
$$(2) \int_{-\infty}^0 x e^x dx = \lim_{t \rightarrow -\infty} \int_t^0 x e^x dx$$

$$\begin{cases} u = x \\ dv = e^x dx \end{cases} \quad \begin{cases} du = dx \\ v = e^x \end{cases}$$

$$= \lim_{t \rightarrow -\infty} \left(x e^x \Big|_t^0 - \int_t^0 e^x dx \right)$$

$$= \lim_{t \rightarrow -\infty} \left(-t e^t - \underbrace{(e^0)}_1 + e^t \right)$$

$$= -1 + \lim_{t \rightarrow -\infty} (-t e^t + e^t)$$



$$= -1 + \lim_{t \rightarrow -\infty} e^t \cdot (1 - t)$$

$$= -1 + \lim_{t \rightarrow -\infty} \frac{1-t}{\frac{1}{e^t}} \quad \text{to use L'Hopital.}$$

$$= -1 + \lim_{t \rightarrow -\infty} \frac{1-t}{e^{-t}}$$

$$= -1 + \lim_{t \rightarrow -\infty} \frac{-1}{-e^{-t}} = -1$$

$$(3) \int_{-\infty}^{\infty} x e^{-x^2} dx = \int_{-\infty}^0 x e^{-x^2} dx + \int_0^{\infty} x e^{-x^2} dx$$

$$\int_{-\infty}^0 x e^{-x^2} dx = \lim_{t \rightarrow -\infty} \left[-\frac{1}{2} \int_t^0 -2x e^{-x^2} dx \right]$$

$$u = -x^2 \\ du = -2x dx$$

$$\int e^u du = e^u$$

$$= \lim_{t \rightarrow -\infty} \left[-\frac{1}{2} \cdot e^{-x^2} \right]_t^0$$

$$= \lim_{t \rightarrow -\infty} -\frac{1}{2} \left[e^0 - e^{-t^2} \right]$$

$$= \lim_{t \rightarrow -\infty} -\frac{1}{2} \left[1 - e^{-t^2} \right]$$

$$= -\frac{1}{2}$$

$$\begin{aligned}\int_0^{\infty} x e^{-x^2} dx &= \lim_{t \rightarrow \infty} \int_0^t x e^{-x^2} dx \\&= \lim_{t \rightarrow \infty} \left(-\frac{1}{2} e^{-x^2} \right) \Big|_0^t \\&= \lim_{t \rightarrow \infty} \left(-\frac{1}{2} \cdot (e^{-t^2} - \underbrace{e^0}_{-1}) \right) \\&= \frac{1}{2}\end{aligned}$$

$$\text{So, } \int_{-\infty}^{\infty} x e^{-x^2} dx = -\frac{1}{2} + \frac{1}{2} = 0$$

Type 2 Improper Integrals:

Integrals of Discontinuous Functions.

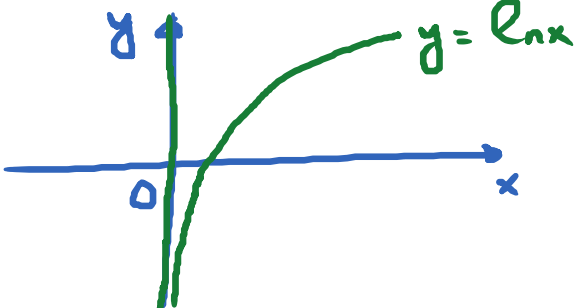
E.g. $\int_{-2}^3 \frac{1}{x^3} dx = \int_{-2}^3 x^{-3} dx = \frac{x^{-2}}{-2} \Big|_{-2}^3$

$= -\frac{1}{2x^2} \Big|_{-2}^3 = -\frac{1}{18} + \frac{1}{8} = \dots$

Wrong.

$\frac{1}{x^3}$ is discontinuous at 0, a point in $[-2, 3]$.

E.g. $\int_0^1 \ln x dx$



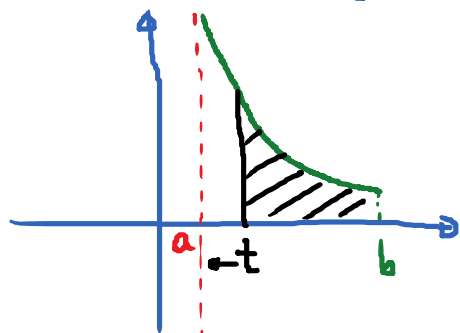
The graph shows the function $y = \ln x$ on a coordinate plane. The x-axis and y-axis are shown, with the origin labeled 0. The curve $y = \ln x$ approaches negative infinity as x approaches 0 from the right, indicating a vertical asymptote at $x = 0$. The curve crosses the x-axis at $x = 1$ and continues to increase as x increases.

E.g. $\int_0^3 \frac{dx}{\sqrt{3-x}}$

$\frac{1}{\sqrt{3-x}}$ has a discontinuity at 3.

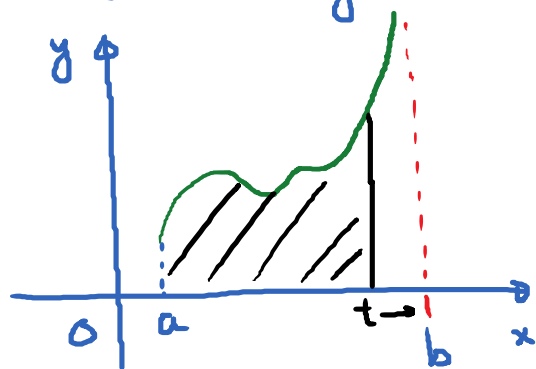
→ None of these are proper integrals.

* Discontinuity at left endpoint. $\int_a^b f(x) dx$, f has a discontinuity at $x=a$.



$$\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$$

* Discontinuity at the right endpoint.

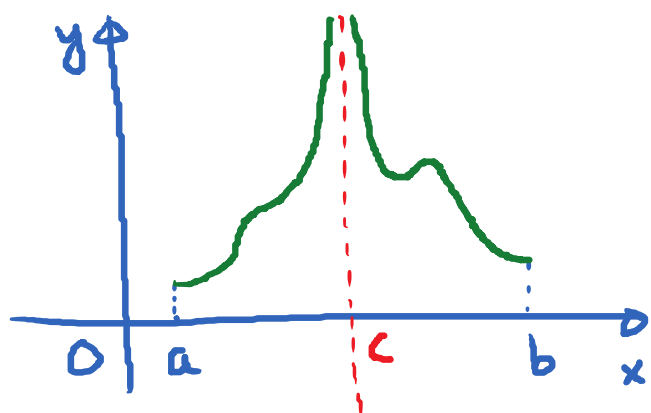


$\int_a^b f(x) dx$, f has a discontinuity at $x=b$.

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$

* Discontinuity at a point c in (a, b) :

f has a discontinuity at c in (a, b)



$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

$$= \lim_{t \rightarrow c^-} \int_a^t f(x) dx + \lim_{t \rightarrow c^+} \int_t^b f(x) dx$$

$\int_a^b f(x) dx$ converges if both of these limits exist and finite.

E.g.

$$\int_{-2}^3 \frac{1}{x^3} dx = \int_{-2}^0 \frac{1}{x^3} dx + \int_0^3 \frac{1}{x^3} dx$$

$$\int_{-2}^0 \frac{1}{x^3} dx = \lim_{t \rightarrow 0^-} \int_{-2}^t \frac{1}{x^3} dx = \lim_{t \rightarrow 0^-} \left(-\frac{1}{2x^2} \right) \Big|_{-2}^t$$

$$= \lim_{t \rightarrow 0^-} \left(-\frac{1}{2t^2} + \frac{1}{8} \right) = -\infty$$

So, $\int_{-2}^3 \frac{1}{x^3} dx$ diverges.

Ex. $\int_0^3 \frac{dx}{\sqrt{3-x}}$ converges or diverges?

$$= \lim_{t \rightarrow 3^-} \int_0^t \frac{dx}{\sqrt{3-x}} = \lim_{t \rightarrow 3^-} - \int_0^t (3-x)^{-1/2} (-dx)$$

$$= \lim_{t \rightarrow 3^-} \left(-2\sqrt{3-x} \right) \Big|_0^t$$

$$= \lim_{t \rightarrow 3^-} \left(-2\sqrt{3-t} + 2\sqrt{3} \right)$$

$$= 2\sqrt{3}. \quad \text{So, } \int_0^3 \frac{dx}{\sqrt{3-x}} \text{ converges to } 2\sqrt{3}.$$

$$u = 3-x$$

$$du = -dx$$

$$\int u^{-1/2} du = -\frac{u^{1/2}}{1/2} = -2\sqrt{u}$$