5.2. Infinite Series

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An infinite series (or just series) is the sum of all the terms of an infinite sequence.

$\{a_1, a_2, a_3, a_4, \ldots\} \longrightarrow$ sequence. $\{a_i\}_{i=1}^{\infty}$

$a_1 + a_2 + a_3 + a_4 + \cdots \longrightarrow$ series.

<u>Notation:</u> $\displaystyle\sum_{i=1}^{\infty} a_i$

<u>E.g.</u> $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots\right\}$. Notation: $\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$

$\longrightarrow$ Sequence.

Form a series: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

$$\longrightarrow \sum_{n=1}^{\infty} \frac{1}{2^n}$$

How do we think about adding up infinitely many terms?

Add first 2 terms: $\frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 0.75 \longrightarrow 2^{\text{nd}}$ partial sum

Add first 3 terms: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 0.875 \longrightarrow 3^{\text{nd}}$ partial sum

4 terms: $\frac{1}{2} + \cdots + \frac{1}{16} = 0.9375 \longrightarrow 4^{\text{th}}$ partial sum

15^{th} partial sum $\longleftarrow$ 15 terms: $\frac{1}{2} + \cdots + \frac{1}{2^{15}} = 0.9999694$

25^{th} partial sum $\longleftarrow$ 25 terms: $\frac{1}{2} + \cdots + \frac{1}{2^{25}} = 0.999999701$

Sum of the infinite series = limit of these partial sums.

To define the sum of an infinite series:

$$S = \sum_{i=1}^{\infty} a_i$$

We consider the sequence of partial sums:

$S_1 = a_1 \longleftarrow 1^{\text{st}}$ partial sum

$S_2 = a_1 + a_2 \longleftarrow 2^{\text{nd}}$ partial sum

$S_3 = a_1 + a_2 + a_3 \longleftarrow 3^{\text{nd}}$ partial sum

$\vdots$

$S_n = a_1 + a_2 + \cdots + a_n \longleftarrow n^{\text{th}}$ partial sum (Sum of first n terms)

→ Sequence of partial sums $\{S_1, S_2, S_3, \dots, S_n, \dots\}$

If $\lim\limits_{n \to \infty} S_n$ exists (as a finite number), then we call the limit the sum of the infinite series.

In other words

$$\boxed{S = \sum_{i=1}^{\infty} a_i := \lim_{n \to \infty} S_n}$$

Back to our problem:

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \lim_{n \to \infty} S_n$$

where S_n is the $n^{\underline{th}}$ partial sum of our series.

$$S_n = \sum_{i=1}^{n} \frac{1}{2^i} = \underbrace{\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n}}_{?}$$

→ try to find a formula in terms of n for S_n and take the limit.

$$\boxed{S_n = \boxed{\frac{1}{2}} + \cancel{\frac{1}{2^2}} + \cancel{\frac{1}{2^3}} + \cdots + \cancel{\frac{1}{2^n}}}$$

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$$\frac{1}{2}\cdot S_n = \frac{1}{2}\cdot\left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n}\right)$$

$$\boxed{\frac{1}{2}\cdot S_n = \cancel{\frac{1}{2^2}} + \cancel{\frac{1}{2^3}} + \cdots + \cancel{\frac{1}{2^n}} + \frac{1}{2^{n+1}}.}$$

$$\boxed{S_n - \frac{1}{2}\cdot S_n} = \frac{1}{2} - \frac{1}{2^{n+1}}.$$

$$\frac{1}{2}\cdot S_n = \frac{1}{2} - \frac{1}{2^{n+1}}$$

$$S_n = 2\cdot\left(\frac{1}{2} - \frac{1}{2^{n+1}}\right) = 1 - \frac{2}{2^{n+1}}$$

$$\boxed{S_n = 1 - \frac{1}{2^n}}$$ $\longrightarrow$ a formula for the n^{th} partial sum of our series

By definition,

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{2^n}\right)$$

$$= 1.$$

So, $\boxed{\sum_{i=1}^{\infty} \frac{1}{2^i} = 1}$

The series we just analyzed is an example of a geometric series.

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$$

multiply by $\frac{1}{2}$ $\times\frac{1}{2}$ $\times\frac{1}{2}$

common ratio $= \frac{1}{2}$.

→ To get from one term to the next, we always multiply by the same constant. That constant is called the $\boxed{\text{common ratio}}$ of the series.

E.g. $s = 5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$

$\times\left(-\frac{2}{3}\right)$ $\times\left(-\frac{2}{3}\right)$ $\times\left(-\frac{2}{3}\right)$

→ this is geometric. Common ratio is $-\frac{2}{3}$.