Key result on geometric series:

If the absolute value of the common ratio is less than 1, the series converges.

Otherwise, it diverges.

If it converges, it will converge to the number:

$$s = \frac{\text{first term of series}}{1 - \text{common ratio}}.$$

$$s = 5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

Common ratio $= -\frac{2}{3}$. $\left|-\frac{2}{3}\right| < 1 \rightarrow$ converges.

Converges to $s = \dfrac{5}{1-\left(-\frac{2}{3}\right)} = \dfrac{5}{1+\frac{2}{3}}$

$$= \frac{5}{\frac{5}{3}} = 5 \cdot \frac{3}{5} = \boxed{3}.$$

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{2-5^n}{10^n}$

Q: Does this series converge?

If it does, what does it converge to?

$$\sum_{n=1}^{\infty}\left(\frac{2}{10^n}-\frac{5^n}{10^n}\right) = \underbrace{\sum_{n=1}^{\infty}\frac{2}{10^n}} - \underbrace{\sum_{n=1}^{\infty}\frac{5^n}{10^n}}$$

$$= \left(\frac{2}{10}+\frac{2}{10^2}+\frac{2}{10^3}+\cdots\right) - \left(\frac{5}{10}+\frac{5^2}{10^2}+\frac{5^3}{10^3}+\cdots\right)$$

$\frac{1}{10}$ $\frac{1}{10}$ $\frac{5}{10}$ $\frac{5}{10}$

$$= \frac{\frac{2}{10}}{1-\frac{1}{10}} - \frac{\frac{1}{2}}{1-\frac{1}{2}}$$

$$= \frac{\frac{2}{10}}{\frac{9}{10}} - \frac{\frac{1}{2}}{\frac{1}{2}} = \frac{2}{9} - 1 = \boxed{-\frac{7}{9}}$$

E.g. $$\sum_{n=3}^{\infty} \frac{(\ln(x))^{n-3}}{6^n}$$

Q: Find the value(s) of x for which this series converges.

$$\sum_{n=3}^{\infty} \frac{(\ln(x))^{n-3}}{6^n} = \frac{1}{6^3} + \frac{\ln(x)}{6^4} + \frac{[\ln(x)]^2}{6^5} + \cdots$$

$$\frac{\ln(x)}{6} \qquad \frac{\ln(x)}{6}$$

$\longrightarrow$ geometric with common ratio $= \frac{\ln(x)}{6}$.

For it to converge, $\left| \frac{\ln(x)}{6} \right| < 1$.

$$-1 < \frac{\ln(x)}{6} < 1$$

$$-6 < \ln(x) < 6$$

$$e^{-6} < x < e^{6}$$

Q: If it converges, what does it converge to?

$$s = \boxed{\frac{\frac{1}{6^3}}{1 - \frac{\ln(x)}{6}}}$$

Telescoping Series:

E.g. $\displaystyle\sum_{i=1}^{\infty} \frac{1}{(i+1)\cdot(i+2)}$.

$$\frac{1}{(i+1)(i+2)} = \frac{A}{i+1} + \frac{B}{i+2}$$

$\longrightarrow$ Solve for A and B: $A = 1$, $B = -1$.

$$\sum_{i=1}^{\infty} \left[\frac{1}{i+1} - \frac{1}{i+2} \right].$$

$$S_1 = \frac{1}{2} - \frac{1}{3}$$

$$S_2 = \left[\frac{1}{2} - \cancel{\frac{1}{3}}\right] + \left[\cancel{\frac{1}{3}} - \frac{1}{4}\right]$$

i = 1 i = 2

$$S_3 = \left[\frac{1}{2} - \cancel{\frac{1}{3}}\right] + \left[\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right] + \left[\cancel{\frac{1}{4}} - \frac{1}{5}\right]$$

i = 1 i = 2 i = 3

$$S_n = \left[\frac{1}{2} - \cancel{\frac{1}{3}}\right] + \left[\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right] + \cdots + \left[\cancel{\frac{1}{n+1}} - \frac{1}{n+2}\right]$$

$$S_n = \frac{1}{2} - \frac{1}{n+2} \longleftarrow$$ formula for n^{th} partial sum.

$$\rightarrow \sum_{i=1}^{\infty} \frac{1}{(i+1)(i+2)} = \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty}\left[\frac{1}{2} - \frac{1}{n+2}\right] = \boxed{\frac{1}{2}}$$