This result tells us that:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} p > 1, \text{ the series converges} \\ p \leq 1, \text{ the series diverges.} \end{cases}$$

p-series.

E.g. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$. This is a p-series when $p = \frac{1}{3}$.

So, it diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n \cdot n^{1/2}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

This is a p-series where $p = \frac{3}{2}$.

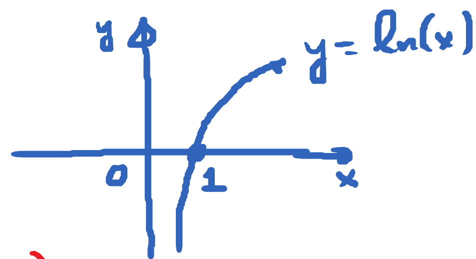
It converges.

E.g. (HW #4)

Does $\displaystyle\sum_{n=2}^{\infty} \frac{7}{n\sqrt{\ln(n)}}$ converge or diverge?

The function which generates the terms of this series is

$$f(x) = \frac{7}{x\sqrt{\ln(x)}}.$$



* Is f continuous on $[2, \infty)$?

  Yes. because f is defined on $[2, \infty)$.

* Is f positive on $[2, \infty)$?

  Yes.

* Is f decreasing on $[2, \infty)$?

  Yes.

All conditions of integral test are satisfied.

$\longrightarrow$ Apply the Integral Test.

$$\int_2^\infty \frac{7}{x\sqrt{\ln x}}\,dx = 7\cdot\int_2^\infty \frac{dx}{x\sqrt{\ln x}}$$

$$= 7\cdot\lim_{t\to\infty}\int_2^t \frac{dx}{x\sqrt{\ln x}}$$

du, u

Let $u = \ln x$. $du = \frac{dx}{x}$

$$\int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du = \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \frac{u^{1/2}}{1/2}$$

$$= 2\sqrt{u}$$

$$7\cdot\lim_{t\to\infty} 2\sqrt{\ln x}\,\Big|_2^t$$

$$= 14\cdot\lim_{t\to\infty}\left(\boxed{\sqrt{\ln t}-\sqrt{\ln 2}}\right) = \infty$$

So, the integral diverges. Hence, the series diverges.

Remainder Estimate from the Integral Test.

Suppose that we have used the integral test to determine that the series $\sum_{n=1}^{\infty} a_n$ converges.

Suppose that we then use a partial sum from $n=1$ to $n=N$: $\boxed{\sum_{n=1}^{N} a_n}$ to approximate the value of this infinite series.

We let $S_N = \sum_{n=1}^{N} a_n$; $s = \sum_{n=1}^{\infty} a_n$.

We want to know how far away S_N is from the infinite sum s.

Let $R_N = s - S_N$

$\longrightarrow$ want bounds for the remainder R_N.

The integral test says that

$$\int_{N+1}^{\infty} f(x)\,dx < R_N < \int_{N}^{\infty} f(x)\,dx$$

a lower bound for the remainder.

an upper bound for the remainder

E.g. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

This series converges because it is a p series where p is $3 > 1$.

Suppose that we use $S_{10} = \sum_{n=1}^{10} \frac{1}{n^3}$ to approximate the sum of this infinite series.

$$S_{10} = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots + \frac{1}{1000}$$

$$\approx 1.1975$$

Q: Estimate the error:

infinite sum

$$R_{10} = s - S_{10}$$

Integral test says that:

$$\int_{11}^{\infty} \frac{1}{x^3}\,dx < R_{10} < \int_{10}^{\infty} \frac{1}{x^3}\,dx$$

0.004 0.005

$$\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \frac{x^{-2}}{-2} = -\frac{1}{2x^2}$$

$$\int_{10}^{\infty} \frac{1}{x^3}\,dx = \left(-\frac{1}{2x^2}\right)\Big|_{10}^{\infty} = 0 + \frac{1}{200} = 0.005$$

$$\int_{11}^{\infty} \frac{1}{x^3}\,dx = \left(-\frac{1}{2x^2}\right)\Big|_{11}^{\infty} = 0 + \frac{1}{242} \simeq 0.004$$

Q: How many terms should we use so that the error is no more than 0.001?

(Find N such that S_N will estimate s to within 0.001)

By the integral test, the error is bounded as follows:

$$\int_{N+1}^{\infty} \frac{1}{x^3}\,dx < R_N < \int_{N}^{\infty} \frac{1}{x^3}\,dx$$

All we need is to require the upper bound to be no more than 0.001, that is, we want

$$\int_{N}^{\infty} \frac{1}{x^3}\,dx < 0.001$$

$$\left(-\frac{1}{2x^2}\right)\Bigg|_{N}^{\infty} < 0.001$$

$$\frac{1}{2N^2} < 0.001$$

$$\rightarrow \frac{1}{N^2} < 0.002 \longrightarrow \frac{1}{0.002} < N^2$$

$$\rightarrow N > \sqrt{\frac{1}{0.002}} \approx 22.36$$

$\longrightarrow N = 23$ will give us the desired accuracy.

That is, $S_{23} = 1 + \frac{1}{8} + \frac{1}{27} + \cdots + \frac{1}{(23)^3}$

will be within 0.001 of the infinite sum.