5.3. The Divergence Test and the Integral Test.

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① The Divergence Test

Suppose we have a series $\sum_{n=1}^{\infty} \boxed{a_n}$

If $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0$, then the series diverges.

formula for the general n^{th} term.

E.g. $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$; $a_n = \frac{n^2}{5n^2+4}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2+4} = \frac{1}{5} \neq 0$$

$\longrightarrow$ the series diverges by the Divergence Test

E.g. $\sum_{n=1}^{\infty} e^{1/n^2}$. $a_n = e^{1/n^2}$.

$1/n^2 \to 0$

$$\lim_{n\to\infty} e^{1/n^2} = e^0 = 1 \neq 0.$$ Series Diverges.

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E.g. $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n^2}\right)$; $a_n = \cos\left(\frac{1}{n^2}\right)$

$\lim_{n\to\infty} \cos\left(\frac{1}{n^2}\right) = \cos(0) = 1 \neq 0.$ ($\frac{1}{n^2} \to 0$)

Series Diverges.

Warning: If $\lim_{n\to\infty} a_n = 0$, the Divergence test fails. We do <u>NOT</u> know whether the series converges or not just based on this.

E.g. $\sum_{n=1}^{\infty} \frac{1}{n}$; $a_n = \frac{1}{n}$ → Harmonic series

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

$\frac{\pi^2}{6}$ ← $\sum_{n=1}^{\infty} \frac{1}{n^2}$; $a_n = \frac{1}{n^2}$; $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$

The Divergence Test fails in both cases. As we will see later, using another test, the first series diverges whereas the second one converges.

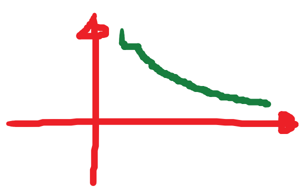
2 The Integral Test.

Suppose $s = \sum_{n=1}^{\infty} a_n$ is a series.

Suppose $a_n = f(n)$.

If all the following conditions are satisfied:

1. f is positive on $[N, \infty)$ for some $N \geq 1$
2. f is continuous on $[N, \infty)$
3. f is decreasing on $[N, \infty)$

( To test for decreasing: $f' < 0$)

Then: $\sum_{n=1}^{\infty} a_n$ and $\int_N^{\infty} f(x)\,dx$

are both convergent or both divergent.

In other words, if $\int_N^{\infty} f(x)\,dx$ converges, our series will converge.

if $\int_N^{\infty} f(x)\,dx$ diverges, our series will diverge.

E.g. 1 Consider $s = \sum_{n=1}^{\infty} \frac{1}{n}$; $f(n) = \frac{1}{n}$

So, $f(x) = \frac{1}{x}$ is the function associated with this series.

1. Is f positive on $[1, \infty)$? Yes.
2. Is f continuous on $[1, \infty)$? Yes.
3. Is f decreasing on $[1, \infty)$? Yes.

$\left(f'(x) = -\frac{1}{x^2} < 0 \right)$

$\longrightarrow$ Now we can use the integral test.

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} \left(\ln|x| \Big|_1^b \right)$$

$$= \lim_{b\to\infty} \left(\ln(b) - \ln(1) \right)$$

$$= \lim_{b\to\infty} \ln(b) = \infty$$

So, the integral diverges.

By the integral test, the original series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Ex. Use the integral test for $\sum_{n=1}^{\infty} \frac{1}{n^2}$. (Verify that all conditions for using the test are satisfied before using it)

The function associated with this series is $f(x) = \frac{1}{x^2}$.

(1) Is f positive on $[1, \infty)$? Yes.

(2) Is f continuous on $[1, \infty)$? Yes.

(3) Is f decreasing on $[1, \infty)$? Yes.

$\longrightarrow$ apply integral test.

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$$

$$= \lim_{b\to\infty} \int_1^b x^{-2}\,dx$$

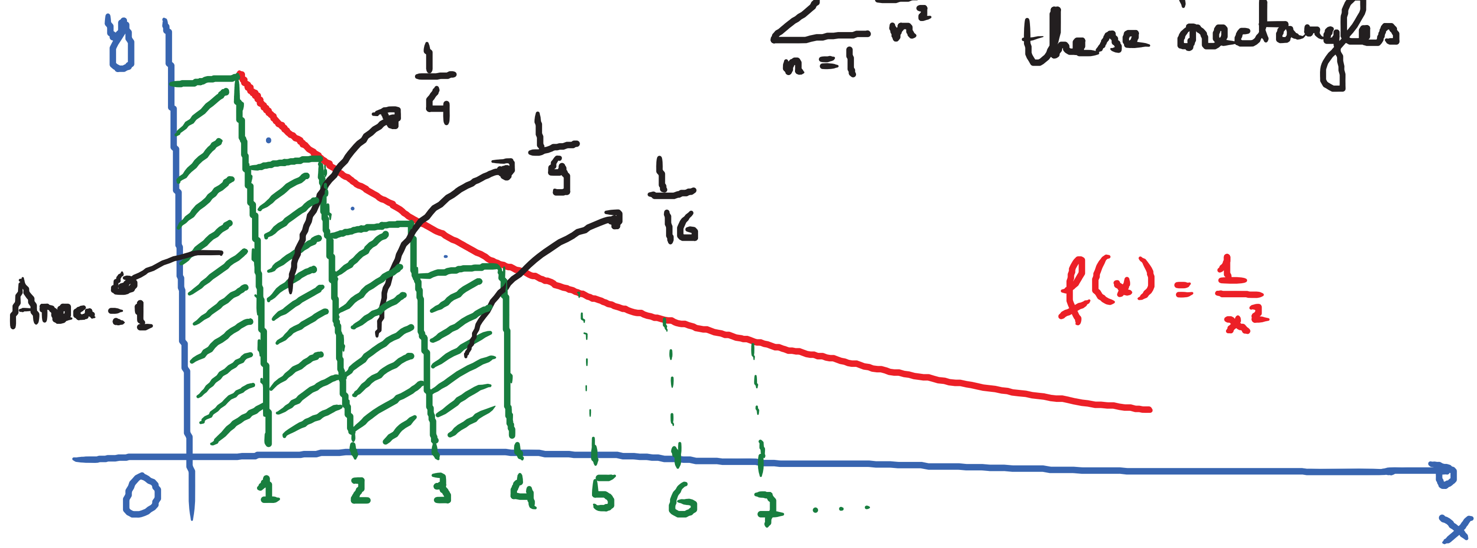
$$= \lim_{b\to\infty} \left(\frac{x^{-1}}{-1} \Big|_1^b \right) = \lim_{b\to\infty} \left(-\frac{1}{x} \Big|_1^b \right)$$

$$= \lim_{b\to\infty} \left(-\frac{1}{b} + 1 \right) = 1$$

The integral converges. Thus, the series converges.

Note: The value of the integral is NOT what this series converges to. The integral only tells us that the series converges, it does not tell us what the series converges to.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \text{Sum of areas of these rectangles}$$



Note: Useful improper integrals to keep in mind.

p-integrals.

$$\int_1^{\infty} \frac{1}{x^p}\,dx$$

if $p > 1$, this integral converges.

if $p \leq 1$, this integral diverges.