5.4. Comparison Tests and Limit Comparison Tests

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Direct Comparison Test.

Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

If $0 \leq a_n \leq b_n$ for all $n \geq N$, then:

① If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

② If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

E.g. Consider $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$. a_n

Does this series converge or diverge?

Idea: Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ b_n → converges b/c it is a p-series with $p = 2 > 1$.

$a_n = \frac{1}{n^2+1}$ and $b_n = \frac{1}{n^2}$. Is $a_n \leq b_n$ for all n?

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$$\frac{1}{n^2+1} \overset{?}{\leq} \frac{1}{n^2} \text{ for all } n \geq 1 \longrightarrow \text{True}$$

So, $$\sum_{n=1}^{\infty} \frac{1}{n^2+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$ converges.

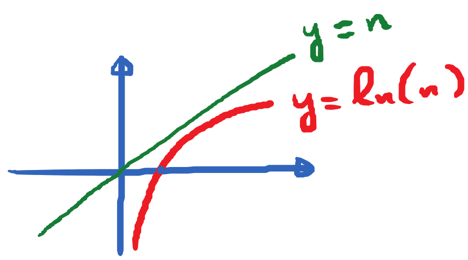
Conclusion: $$\sum_{n=1}^{\infty} \frac{1}{n^2+1} \text{ must converge.}$$

Ex. Determine whether the given series converges or diverges by using the direct comparison test.

① $$\sum_{n=1}^{\infty} \frac{1}{n^3+3n+1}$$

② $$\sum_{n=1}^{\infty} \frac{1}{2^n+1}$$

③ $$\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$$



Sol. ① Compare it with $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges b/c it is a p-series with $p = 3 > 1$.

$n^3 + 3n + 1 > n^3$ for all $n \geq 1$

So, $\frac{1}{n^3+3n+1} < \frac{1}{n^3}$ for all $n \geq 1$.

So, $\sum_{n=1}^{\infty} \frac{1}{n^3+3n+1} < \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges

So, $\sum_{n=1}^{\infty} \frac{1}{n^3+3n+1}$ converges b/c it is "smaller" than a convergent series.

② Compare with $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

We know $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$

$\times \frac{1}{2}$ $\times \frac{1}{2}$ $\times \frac{1}{2}$

It is a geometric series with common ratio $= \frac{1}{2} < 1$.

So, $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges.

$2^n + 1 > 2^n$ for all $n \geq 1$.

So, $\frac{1}{2^n+1} < \frac{1}{2^n}$

converges

$$\sum_{n=1}^{\infty} \frac{1}{2^n+1} < \sum_{n=1}^{\infty} \frac{1}{2^n}$$

So, $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ converges.

diverges

3) $\ln(n) < n$ for all $n \geq 2$.

$$\frac{1}{\ln(n)} > \frac{1}{n}. \qquad \sum_{n=2}^{\infty} \frac{1}{\ln(n)} > \sum_{n=2}^{\infty} \frac{1}{n}$$

p-series
$p = 1$

So, $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ diverges because it is "larger" than a divergent series.

Limit Comparison Test

Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

Limit comparison test says that:

If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ where L is a finite and positive number; i.e., $L > 0$.

then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ will behave in the same way.

In other words, if one converges, so does the other. If one diverges, so does the other.

E.g. $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{n^5+5}}$

Scratch work: when n is large, the series behaves like: $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2}{\sqrt{n^5}} = \sum_{n=1}^{\infty} \frac{2n^2}{n^{5/2}} = \sum_{n=1}^{\infty} \frac{2}{n^{1/2}}$

diverges $p = 1/2$

But we can't (it is hard!) directly compare the two. $\longrightarrow$ So, limit comparison test.

Real work:

$$\lim_{n\to\infty} \frac{\dfrac{2n^2+3n}{\sqrt{n^5+5}}}{\dfrac{2}{\sqrt{n}}} = \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{n^5+5}} \cdot \frac{\sqrt{n}}{2}$$

$$\lim_{n\to\infty} \frac{(2n^2+3n)\sqrt{n}}{2\sqrt{n^5+5}} = 1 > 0$$

So, $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{n^5+5}}$ behaves exactly like

$\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}}$ by the limit comparison test.

Since $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}}$ diverges b/c it is a p-series with $p = 1/2$.

The original series must diverge.

Ex. Determine whether the series converges or diverges.

① $\displaystyle\sum_{n=1}^{\infty} \frac{n^4+2n^2-1}{6n^6+4n^4}$ ② $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$

Limit comparing with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{6n^2}$ $\displaystyle\sum \frac{\sqrt{n}}{n} = \sum \frac{1}{\sqrt{n}}$

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③ $\sum_{n=1}^{\infty} \frac{\ln\left(1+\frac{4}{n}\right)}{n}$.

④ $\boxed{\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^2}} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$

Converges

Converges