5.5. Alternating Series

Recall: The Integral Test and the comparison test only apply to series with positive terms.

E.g. $\boxed{\sum_{n=1}^{\infty} \frac{1}{n}}$ → Integral Test — Diverges

$\boxed{\sum_{n=1}^{\infty} \frac{n^2}{2n^3+1}}$ Integral test / Limit Comparison Test } Diverges.

What the series has negative terms? alternating series

$$\sum_{n=1}^{\infty} (-1)^n \cdot \boxed{\frac{1}{n}} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \cdots$$

$b_n = \frac{1}{n}$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \boxed{\frac{n^2}{2n^3+1}} = +\frac{1}{3} - \frac{4}{17} + \frac{9}{55} - \cdots$$

$b_n = \frac{n^2}{2n^3+1}$

Alternating Series Test

Given an alternating series $\boxed{\sum_{n=1}^{\infty} (-1)^n b_n}$ or $\boxed{\sum_{n=1}^{\infty} (-1)^{n+1} b_n}$; with $b_n > 0$.

If the sequence $\{b_n\}$ satisfies the conditions:

(1) $b_n \geq b_{n+1}$ for all n. (the terms are getting smaller)

(2) $\lim_{n\to\infty} b_n = 0$

Then the alternating series converges.

E.g. $\sum_{n=1}^{\infty} (-1)^n \cdot \boxed{\frac{1}{n}}$ → b_n

$b_n = \frac{1}{n}$ (1) Is $b_n \geq b_{n+1}$ for all n? Yes

$\frac{1}{n} \geq \frac{1}{n+1}$ for all n

(2) Is $\lim_{n\to\infty} b_n = 0$? Yes

$\lim_{n\to\infty} \frac{1}{n} = 0$

And the series is an alternating series.

Therefore, by the Alternating Series Test, it converges.

E.g. $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \boxed{\frac{n^2}{2n^3+1}}$ → Alternating Series.

b_n

(2) Is $\lim_{n\to\infty} b_n = 0$?

$\lim_{n\to\infty} \frac{n^2}{2n^3+1} = 0$

$\frac{n^2}{2n^3} = \frac{1}{2n} \to 0$

$\frac{\infty}{\infty}$ limit

L'Hopital

$= \lim_{n\to\infty} \frac{2n}{6n^2} = \lim_{n\to\infty} \frac{1}{3n} = 0$

(1) Is $b_n \geq b_{n+1}$ for all n? ✓

$b_n = f(n) = \frac{n^2}{2n^3+1}$.

If we can show that f is decreasing, then b_n is a decreasing sequence. → Find $f'(n)$.

$$f(n) = \frac{n^2}{2n^3+1} \longrightarrow f'(n) = \frac{(2n^3+1)\cdot 2n - n^2\cdot 6n^2}{(2n^3+1)^2}$$

$$= \frac{4n^4 + 2n - 6n^4}{(2n^3+1)^2} \qquad \leq 0$$

$$= \frac{-2n^4+2n}{(2n^3+1)^2} = \frac{2n(1-n^3)}{(2n^3+1)^2}$$

So, $f'(n) \leq 0$ on $[1,\infty)$

So, b_n is decreasing

Done! Series converges by AST.

Ex. Determine whether the given series converges or diverges.

① $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n!}$

($n! = n\cdot(n-1)(n-2)(n-3)\cdots 1$

n factorial

E.g. $5! = 5\cdot4\cdot3\cdot2\cdot1 = 120$

$4! = 4\cdot3\cdot2\cdot1 = 24$

$1! = 1$; $0! := 1$)

② $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{2n^2}{n^2+1}$

③ $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \frac{\ln(n)}{n^2}$

④ $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \cos\left(\frac{\pi}{n}\right)$.

Sol:

① $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$

$b_n = \frac{1}{n!}$

$\frac{1}{n!} \geq \frac{1}{(n+1)!}$

Why is $b_n \geq b_{n+1}$? ✓

Is $\displaystyle\lim_{n\to\infty} \frac{1}{n!} = 0$ ✓

$\longrightarrow$ Converges by AST.

② $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n^2}{n^2+1}$

$\displaystyle\lim_{n\to\infty} \frac{2n^2}{n^2+1} = 2 \neq 0$. So, $\displaystyle\lim_{n\to\infty} (-1)^{n+1}\cdot\frac{2n^2}{n^2+1} \neq 0$

So series diverges by the Divergence Test.

③ $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \boxed{\frac{\ln(n)}{n^2}}$ $\quad b_n = \frac{\ln(n)}{n^2}$

* $\displaystyle\lim_{n\to\infty} \frac{\ln(n)}{n^2} \left(\frac{\infty}{\infty}\right) = \lim_{n\to\infty} \frac{\frac{1}{n}}{2n} = \lim_{n\to\infty} \frac{1}{2n^2} = 0$

* $f(n) = \frac{\ln(n)}{n^2} \longrightarrow f'(n) = \frac{n^2\cdot\frac{1}{n} - \ln(n)\cdot 2n}{n^4}$

$$= \frac{n - 2n\ln(n)}{n^4}$$

$$= \frac{1-2\ln(n)}{n^3} \leq 0 \text{ on } [2,\infty)$$

So, b_n will eventually be decreasing. (Starting at $n=2$)

$\longrightarrow$ Converges by AST.