(4) $\sum_{n=1}^{\infty} (-1)^n \cdot \cos\left(\frac{\pi}{n}\right)$

$\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos(0) = 1 \neq 0$

$\frac{\pi}{n} \to 0$

Diverges by the Divergence test.

Concepts of Absolute Convergence vs. Conditional convergence.

Consider the series $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n} = -1 + \frac{1}{2} - \frac{1}{3} \cdots$

This series converges by AST.

The absolute value series of this is the series

$\sum_{n=1}^{\infty} \frac{1}{n}$. It diverges (b/c p-series, p = 1).

$\longrightarrow$ We say that the original series $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$ converges conditionally. (i.e. the series itself converges but the absolute value series diverges)

On the other hand, consider $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n^2}$

This series converges by the AST.

The absolute value series of this series is :

$\sum_{n=1}^{\infty} \frac{1}{n^2}$. It converges (b/c p-series p = 2)

We say that the original series converges absolutely (i.e., both the series and its absolute value series converge)

Def: If both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$ converge, then we say that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, then we say that $\sum_{n=1}^{\infty} a_n$ converges conditionally.

Error Estimates for alternating series

Given an alternating series $\sum_{n=1}^{\infty} (-1)^n b_n$.

Suppose that it converges by the AST.

Let say it converges to a number s.

$$s = \sum_{n=1}^{\infty} (-1)^n b_n.$$

Suppose we add up the first N terms to estimate s.

$$S_N = \sum_{n=1}^{N} (-1)^n b_n$$

S_N is a finite sum which is approximation for the infinite sum s. How far away is S_N from s? $\longrightarrow$ An upper bound for $|s - S_N|$

AST says that

$$|s - S_N| \leq b_{N+1}$$

E.g. $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$ $\quad \left(b_n = \frac{1}{n} \right)$

$$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n} = \boxed{-\ln(2)}$$ actual sum

$$s = -\ln(2)$$

$$S_{10} = \sum_{n=1}^{10} (-1)^n \cdot \frac{1}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots + \frac{1}{10} \approx -0.64563$$

$$|s - S_{10}| \leq \frac{1}{11}$$

$$\underbrace{|-\ln(2) - (-0.64563)|}_{\approx 0.04752} \leq \underbrace{\frac{1}{11}}_{0.09}$$

Ex. How many terms should I use in a finite partial sum to approximate the series

$$\sum_{n=1}^{\infty} (-1)^n \cdot \boxed{\frac{1}{n^2}}$$ so that the error is no more that 0.001?

Suppose we use N terms to approximate s.

By AST, $|s - S_N| \leq b_{N+1}$ upper bound for error

All we need is to require $b_{N+1} \leq 0.001$

$$\frac{1}{(N+1)^2} \leq 0.001$$

$$1 \leq (N+1)^2 \cdot (0.001)$$

$$\rightarrow \frac{1}{0.001} \leq (N+1)^2$$

$$1000 \leq (N+1)^2$$

$$\sqrt{1000} \leq N+1$$

$$31.6 \leq N+1$$

$$N \geq 30.6 \longrightarrow$$ we need 31 terms.