5.6. Ratio Test and Root Test

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Ratio Test

Given a series $\sum_{n=1}^{\infty} a_n$

Form the limit $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$

let $\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$

Case 1: $0 \leq \rho < 1$, then the series converges.

And it converges absolutely, i.e., the absolute value series converges.

Case 2: $\rho > 1$, then the series diverges.

Case 3: $\rho = 1$. The test fails; i.e., it does not provide any information about the series. We have to use other tests or techniques.

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E.g. Consider the series: $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n^3}{3^n}$.

Q: Converges or Diverges?

$\longrightarrow$ Apply the Ratio Test. Find $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$a_n = \frac{(-1)^n \cdot n^3}{3^n} \; ; \quad a_{n+1} = \frac{(-1)^{n+1} \cdot (n+1)^3}{3^{n+1}}.$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-1)^{\cancel{n}+1} \cdot (n+1)^3}{3^{\cancel{n}+1}} \cdot \frac{\cancel{3^n}}{\cancel{(-1)^n} \cdot n^3} \right|$$

$$= \left| \frac{(-1) \cdot (n+1)^3}{3n^3} \right| = \frac{(n+1)^3}{3n^3}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)^3}{3n^3} = \boxed{\frac{1}{3}} < 1$$

(deg top = deg bottom = 3. limit = $\frac{\text{leading coeff top}}{\text{leading coeff. bottom}}$

→ Conclusion: the series: $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n^3}{3^n}$ converges absolutely.

E.g. Consider the series: $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot (n!)^2}{(2n)!}$

Recall: $n! = n \cdot (n-1) \cdot (n-2) \cdots 1$

→ Apply Ratio Test. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$a_n = \frac{(-1)^n \cdot (n!)^2}{(2n)!} \; ; \quad a_{n+1} = \frac{(-1)^{n+1} \cdot [(n+1)!]^2}{[2(n+1)]!}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-1)^{n+1} \cdot [(n+1)!]^2}{(2n+2)!} \cdot \frac{(2n)!}{(-1)^n \cdot (n!)^2} \right|$$

$$= \left| \frac{(-1) \cdot (n+1) \cdot (n+1)}{(2n+2)(2n+1)} \right| = \frac{(n+1)^2}{(2n+2)(2n+1)}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{1}{4} < 1$$

So, the series converges absolutely.

Ex. Given the series $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{n!}$

Apply the Ratio Test to test for convergence.

$$a_n = \frac{n^n}{n!}\ ; \quad a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{\cancel{n!}}{n^n} = \frac{(n+1)^{n+1}}{(n+1)^1\cdot n^n}$$

$$= \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

$$= \boxed{e} > 1$$

n	$\boxed{\left(1+\frac{1}{n}\right)^n}$
10000	
100000	
1000000	