Conclusion: series diverges.

Why is $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$?

Let $L = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}$.

$n \to \infty$

$1 + \frac{1}{n} \to 1$

This limit has the form: $1^\infty \longrightarrow$ Indeterminate form.

Take ln of both sides:

$$\ln L = \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n$$

$$\ln L = \lim_{n\to\infty} n \cdot \ln\left(1+\frac{1}{n}\right)$$

$$\ln L = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}$$

$\ln\left(1+\frac{1}{n}\right) \longrightarrow 0$

$\frac{1}{n} \longrightarrow 0$

Has $\frac{0}{0}$ form $\longrightarrow$ we can apply L'Hopital.

$$\ln L = \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n}} \cdot \left(\cancel{-\frac{1}{n^2}}\right)}{\cancel{-\frac{1}{n^2}}}$$

$$\ln L = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}$$

$$\ln L = 1 \longrightarrow \boxed{L = e}$$

Note: Have factorial in the series $\rightarrow$ think about the ratio test.

Root Test

Given a series $\sum_{n=1}^{\infty} a_n$.

Form the limit : $\lim_{n\to\infty} \sqrt[n]{|a_n|}$

let $p = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.

Case 1: $0 \le p < 1$: the series converges absolutely

Case 2: $p > 1$, the series diverges.

Case 3: $p = 1$. The test fails, i.e., it does not provide any information about the series.

E.g. Consider the series

$$\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2} \right)^n$$

Q: Converges? Diverges?

$\longrightarrow$ Apply the root test.

$\longrightarrow \lim\limits_{n \to \infty} \sqrt[n]{|a_n|}$

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left|\frac{2n+3}{3n+2}\right|^n} = \frac{2n+3}{3n+2}$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{2n+3}{3n+2} = \boxed{\frac{2}{3}} < 1$$

$\longrightarrow$ The series converges absolutely.

<u>Ex.</u> Consider the series $\displaystyle\sum_{n=2}^{\infty} \frac{n^n}{[\ln(n)]^n}$

Apply the root test to determine whether the series converges or diverges.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{\ln(n)}\right)^n}$$

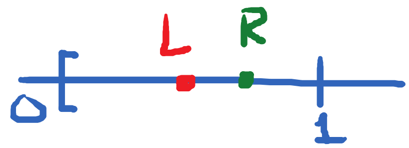
$$= \lim_{n\to\infty} \frac{n}{\ln(n)} \left(\frac{\infty}{\infty}\right) = \lim_{n\to\infty} \frac{1}{\frac{1}{n}} = \lim_{n\to\infty} (n)$$

$$= \boxed{\infty} > 1$$

Series Diverges

Why does the root test work?

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$



Suppose $0 \leq L < 1$.

We will show that the series converges absolutely.

There exists a number R with $L < R < 1$

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} < R$$

This implies that

$\sqrt[n]{|a_n|} < R$ when n is sufficiently large

$\sqrt[n]{|a_n|} < R$ for every $n \geq N$

$|a_n| < R^n$.

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$$\underbrace{\sum_{n=N}^{\infty} |a_n|}_{\text{this must converge}} < \underbrace{\boxed{\sum_{n=N}^{\infty} R^n}}_{\text{this is a geometric series with common ratio } R<1. \text{ So, it converges}}$$

this must converge

this is a geometric series with common ratio $R < 1$.
So, it converges