# 6.1. Power Series

Tuesday, April 3, 2018 1:09 PM

What is a power series?

A power series (centered at 0) is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where $c_n$ are constants.

E.g. If $c_0 = c_1 = c_2 = c_3 = c_4 = \cdots = 1$, i.e., $c_n = 1$ for all $n \geq 0$, then the series $\sum_{n=0}^{\infty} c_n x^n$ looks like:

$$\sum_{n=0}^{\infty} x^n = 1 \xrightarrow{\cdot x} x \xrightarrow{\cdot x} x^2 \xrightarrow{\cdot x} x^3 \xrightarrow{\cdot x} x^4 + \cdots$$

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

This is a geometric series with common ratio $x$.

For this series to converge $|x| < 1$; i.e., $x$ belongs to $(-1, 1)$

Suppose $|x| < 1$ :

$$\boxed{\sum_{n=0}^{\infty} x^n} = \frac{1}{1-x}.$$

Partial sums : $S_1 = \sum_{n=0}^{1} x^n = 1 + x$

$S_2 = \sum_{n=0}^{2} x^n = 1 + x + x^2$

$S_3 = 1 + x + x^2 + x^3$

$S_4 = 1 + x + x^2 + x^3 + x^4$

→ $S_1, S_2, S_3, S_4$, etc ... get close to the function $f(x) = \frac{1}{1-x}$. on $(-1,1)$

---

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \; ; \quad |x| < 1$$

E.g. Consider the power series

$$\sum_{n=0}^{\infty} n!\, x^n \qquad (c_n = n!)$$

$(= 1 + x + 2x^2 + 6x^3 + 24x^4 + \ldots)$

Q: For what values of $x$ does this series converge?

$\longrightarrow$ Apply the Ratio Test

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!\, x^{n+1}}{n!\, x^n}\right| = \left|(n+1)x\right| = (n+1)|x|$$

$$\left(a_n = n!\, x^n \;;\; a_{n+1} = (n+1)!\, x^{n+1}\right)$$

Find $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}(n+1)|x| = \infty > 1$

The series diverges regardless of what $x$ is.

Except for $x = 0$, in which case, series $= 0$.

<u>Conclusion</u>: $x = 0$ is the only value of $x$ for which series converges.

---

<u>Ex.</u> Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{4^n} \cdot x^n . \qquad \left( c_n = \frac{(-1)^n \cdot n}{4^n} \right)$$

<u>Q:</u> Determine the values of $x$ for which the series converges? (Hint: use Ratio Test)

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-1)^{n+1} \cdot (n+1) \cdot x^{n+1}}{4^{n+1}} \cdot \frac{4^n}{(-1)^n \cdot n \cdot x^n} \right|$$

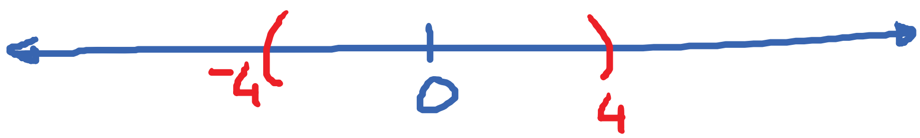
$$= \left| \frac{(-1) \cdot (n+1) \cdot x}{4 \cdot n} \right| = \frac{n+1}{4n} \cdot |x| \quad \text{Ratio}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{n+1}{4n}\cdot|x|\right) = \frac{|x|}{4}$$

For series to converge, we need this limit $< 1$

So, we need $\frac{|x|}{4} < 1$

$$|x| < 4$$




So, x belongs to the interval $(-4, 4)$.

Note: We still need to check whether the series converges or diverges when x equals to the endpoints.

Check $x = 4$: Series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{\cancel{4^n}} \cdot \cancel{4^n}$

$$\sum_{n=1}^{\infty} (-1)^n \cdot n \qquad \lim_{n\to\infty} b_n \neq 0.$$

$b_n$

$(-1)^n \cdot \cancel{4^n}$

So, it diverges by the divergence test.

Check $x = -4$: Series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{\cancel{4^n}} \cdot (-4)^n$

$$= \sum_{n=1}^{\infty} n$$

This series diverges.

Final Conclusion: the values of $x$ for which series converges are $-4 < x < 4$ or $(-4, 4)$

E.g. Consider the power series:

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}. \quad \left( = \sum_{n=1}^{\infty} \frac{1}{n} (x-3)^n \right)$$

$c_n$

Q: For what values of $x$ does it converge?

$\rightarrow$ This is an example of a series centered at 3.

**Note:** $\sum_{n=0}^{\infty} c_n x^n \longrightarrow$ centered at 0

$\sum_{n=0}^{\infty} c_n (x-a)^n \longrightarrow$ centered at a.

**Sol:** Ratio Test again:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right|$$

$$= \left|\frac{(x-3)\cdot n}{n+1}\right| = \frac{n}{n+1} \cdot |x-3|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\underbrace{\frac{n}{n+1}}_{1}|x-3|\right) = |x-3|$$

For the series to converge, we want limit $< 1$.

So, we want: $|x-3| < 1$

$\longrightarrow \quad -1 < x-3 < 1$

$\longrightarrow \quad 2 < x < 4$.

→ Check endpoints.

Check $x = 2$: The series becomes $\sum_{n=1}^{\infty} \frac{(2-3)^n}{n}$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

This series converges by the AST.

Check $x = 4$: The series becomes $\sum_{n=1}^{\infty} \frac{(4-3)^n}{n}$

$$= \sum_{n=1}^{\infty} \frac{1}{n}$$

This series diverges b/c it is a p-series with $p = 1$.

Final conclusion: The series converges for values of $x$ in $[2,4)$ → this is called the interval of convergence for the series