6.2. Properties of Power Series

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Goals: (1) Add, subtract, multiply power series.

(2) Differentiate and Integrate power series.

Recall:

Power series : $\sum_{n=0}^{\infty} c_n x^n$; centered at 0

$\sum_{n=0}^{\infty} c_n (x-a)^n$; centered at a.

Recall that any function that can be written in the form $\frac{1}{1-(\text{Stuff})}$ corresponds to the series

$$\frac{1}{1-(\text{Stuff})} = \sum_{n=0}^{\infty} (\text{Stuff})^n,$$ provided that

$|\text{Stuff}| < 1$.

E.g. $f(x) = \dfrac{3x}{1+x^2} = 3x \cdot \left(\dfrac{1}{1+x^2} \right)$

So, $f(x) = 3x \cdot \left(\underbrace{\dfrac{1}{1-(-x^2)}}_{\sum_{n=0}^{\infty} (-x^2)^n} \text{Stuff} \right)$

$\sum_{n=0}^{\infty} (-x^2)^n$; $\underbrace{|-x^2| < 1}$

$|x|^2 < 1$

$|x| < 1$

$-1 < x < 1$

$f(x) = 3x \cdot \sum_{n=0}^{\infty} (-x^2)^n$

$= 3x \cdot \sum_{n=0}^{\infty} (-1)^n x^{2n}$

$= \sum_{n=0}^{\infty} 3 \cdot (-1)^n \cdot x^{2n+1}$

Conclusion:

$$\boxed{\dfrac{3x}{1+x^2} = \sum_{n=0}^{\infty} 3(-1)^n x^{2n+1}; \quad -1 < x < 1}$$

E.g. $f(x) = \dfrac{x^2}{8 - x^3} = x^2 \cdot \left(\dfrac{1}{8 - x^3} \right)$

$$= x^2 \cdot \left(\frac{1}{8\left(1 - \frac{x^3}{8}\right)} \right)$$

$$= \frac{x^2}{8} \cdot \frac{1}{1 - \frac{x^3}{8}} \quad \text{Stuff}$$

$$= \frac{x^2}{8} \cdot \sum_{n=0}^{\infty} \left(\frac{x^3}{8}\right)^n ; \quad \underbrace{\left|\frac{x^3}{8}\right| < 1}$$

$$= \frac{x^2}{8} \cdot \sum_{n=0}^{\infty} \frac{x^{3n}}{8^n} \qquad |x|^3 < 8$$

$$|x| < 2$$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{x^{3n+2}}{8^{n+1}} ; \quad -2 < x < 2}$$

Add / Subtract Power Series

Suppose $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ are series

with

$$\sum_{n=0}^{\infty} c_n x^n = f(x) \text{ and } \sum_{n=0}^{\infty} d_n x^n = g(x)$$

for x in an interval I

Then:

$$\underbrace{f(x) \pm g(x)}_{h(x)} = \sum_{n=0}^{\infty} c_n x^n \pm \sum_{n=0}^{\infty} d_n x^n$$

$$= \sum_{n=0}^{\infty} (c_n \pm d_n) x^n \text{ for all } x \text{ in } I$$

E.g. Find the power series centered at 0 for the function: $f(x) = \dfrac{7x-1}{3x^2+2x-1}$.

Find the interval of convergence.

$\longrightarrow$ Partial Fractions Decomposition

$$\frac{7x-1}{(3x-1)\cdot(x+1)} = \frac{A}{3x-1} + \frac{B}{x+1}.$$

$$7x-1 = A(x+1) + B(3x-1)$$

Plug $x=-1$ to both sides:

$$-8 = -4B \longrightarrow \boxed{B=2}.$$

Plug $x = \frac{1}{3}$ to both sides:

$$\frac{4}{3} = \frac{4}{3}A \longrightarrow \boxed{A=1}$$

$$f(x) = \frac{1}{3x-1} + \frac{2}{x+1}$$

Now, we find power series for $\frac{2}{x+1}$ and $\frac{1}{3x-1}$ and add them.

$$\frac{2}{1+x} = \frac{2}{1-(-x)} = 2 \cdot \sum_{n=0}^{\infty} (-x)^n \; ; \; |-x| < 1$$

$$|x| < 1$$

$$-1 < x < 1$$

$$\frac{2}{1+x} = \sum_{n=0}^{\infty} 2 \cdot (-1)^n x^n \; ; \; -1 < x < 1$$

$$\frac{1}{3x-1} = \frac{-1}{1-3x} = -\sum_{n=0}^{\infty} (3x)^n \; ; \; |3x| < 1$$

$$|x| < \frac{1}{3}$$

$$= -\sum_{n=0}^{\infty} 3^n x^n \qquad -\frac{1}{3} < x < \frac{1}{3}$$

$$f(x) = \frac{2}{1+x} + \frac{1}{3x-1} = \sum_{n=0}^{\infty} 2 \cdot (-1)^n \cdot x^n - \sum_{n=0}^{\infty} 3^n \cdot x^n$$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \underbrace{\left[2\cdot(-1)^n - 3^n\right]}_{} \cdot x^n}$$

coeff. for power series of f.

Interval of convergence is $\left(-\frac{1}{3}, \frac{1}{3}\right)$

Ex. Find the power series centered at 0 for the function $g(x) = \dfrac{x-4}{x^2-5x+6}$.

Find the I.O.C.

$$\frac{x-4}{x^2-5x+6} = \frac{A}{x-2} + \frac{B}{x-3}$$

$A = 2$ $B = -1$

$$g(x) = \frac{2}{x-2} - \frac{1}{x-3}$$

$$\frac{2}{x-2} = \frac{-2}{2-x} = \frac{-2}{2\left(1-\frac{x}{2}\right)} = \frac{-1}{1-\frac{x}{2}} \quad \text{Stuff}$$

$$= -\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \boxed{-\sum_{n=0}^{\infty}\frac{x^n}{2^n}};$$

$$\left|\frac{x}{2}\right| < 1;\ -2<x<2$$

$$\frac{1}{x-3} = \frac{-1}{3-x} = \frac{-1}{3\left(1-\frac{x}{3}\right)} = -\frac{1}{3}\cdot\frac{1}{1-\frac{x}{3}}$$

$$= -\frac{1}{3}\cdot\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \boxed{-\sum_{n=0}^{\infty}\frac{x^n}{3^{n+1}}};$$

$$-3<x<3$$

$$g(x) = \frac{2}{x-2} - \frac{1}{x-3} = -\sum_{n=0}^{\infty}\frac{x^n}{2^n} + \sum_{n=0}^{\infty}\frac{x^n}{3^{n+1}}$$

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{2^n}+\frac{1}{3^{n+1}}\right)x^n$$

I.O.C. $(-2,2)$.

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Multiply Power Series.

E.g. $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$g(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

Q: Find the coefficients for $x^0, x, x^2, x^3, x^4 \ldots$ in the power series for $f(x) \cdot g(x)$?

$$f(x) \cdot g(x) = \underbrace{\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)}_{f(x)} \underbrace{\left(1 + x + x^2 + x^3 + x^4 + \cdots\right)}_{g(x)}$$

$$= 1 + \underbrace{(1+1)}_{2} x + \underbrace{\left(1 + 1 + \tfrac{1}{2}\right)}_{\frac{5}{2}} x^2$$

$$\left(1 + 1 + \tfrac{1}{2} + \tfrac{1}{6}\right) x^3 + \frac{65}{24} x^4$$