Ex. Suppose that $f(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots$$

Find the coeff. for x^2, x^4 and x^6 in the power series of $[f(x)]^2$.

Term-by-term differentiation and integration of power series.

$f(x) = \sum_{n=0}^{\infty} c_n x^n$ for x in an interval I.

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots.$$

$$\underbrace{f'(x)}_{\frac{d}{dx}(f(x))} = c_1 + 2c_2 x + 3c_3 x^2 + \cdots + n c_n x^{n-1} + \cdots$$

$$= \sum_{n=1}^{\infty} n c_n x^{n-1}.$$ I.O.C. is still I.

If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ on I, then

$$f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} \quad \text{on } I$$

If the series is center at a: $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$,

then $f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$.

For integration:

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \quad \text{on } I.$$

Then $\int f(x)\,dx = \int \left(\sum_{n=0}^{\infty} c_n x^n \right) dx$

$$\longrightarrow \sum_{n=0}^{\infty} \int c_n x^n \, dx = \sum_{n=0}^{\infty} c_n \int x^n \, dx$$

$$= \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}$$

So, If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ on I, then

$$\int f(x)\, dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1} \quad \text{on } I.$$

(similar for center $= a$)

<u>E.g.</u> $f(x) = \dfrac{1}{1-x}$

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \; ; \quad |x| < 1$$

<u>Q:</u> Find the power series (center $= 0$) for

$$g(x) = \frac{1}{(1-x)^2}$$

$$\left[\left(1-x\right)^{-1}\right]' = -1\cdot(1-x)^{-2}\cdot(-1)$$

Thursday, April 5, 2018 2:35 PM

Note: $f'(x) = \left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2}$.

So, $g(x) = f'(x)$

So, to obtain the power series for g, we just need to take the derivative of the power series for f.

$$f(x) = \sum_{n=0}^{\infty} x^n \quad ; \quad |x| < 1$$

$$f'(x) = \sum_{n=1}^{\infty} n x^{n-1} \quad ; \quad |x| < 1$$

$$\text{So, } g(x) = \sum_{n=1}^{\infty} n x^{n-1} \quad ; \quad |x| < 1$$

$$\boxed{\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \quad ; \quad -1 < x < 1}$$

Thursday, April 5, 2018 2:40 PM

$$\left(\frac{1}{(1-x)^2}\right)' = \left[(1-x)^{-2}\right]' = -2(1-x)^{-3}\cdot(-1)$$

$$= \frac{2}{(1-x)^3}.$$

$$\text{So } \frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)\,x^{n-2}$$

$$\text{So, } \frac{1}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2}\cdot x^{n-2} \;;\; -1<x<1$$

$$\underline{\text{E.g.}}\quad f(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n x^n \;;\; -1<x<1$$

$$\text{So, } \frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n \;;\; -1<x<1$$

$$\rightarrow \boxed{\int \frac{1}{1+x}\,dx} = \sum_{n=0}^{\infty}(-1)^n \int x^n\,dx \;;\; -1<x<1$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{n+1}}{n+1} + C$$

$-1 < x < 1.$

Plug $x=0$ to both sides:

$$\ln(1) = C \longrightarrow C = 0$$

So,

$$\boxed{\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \; ; \; -1 < x < 1}$$

E.g. $f(x) = \dfrac{1}{1+x^2} = \dfrac{1}{1-(-x^2)} = \displaystyle\sum_{n=0}^{\infty} (-x^2)^n \; ; \; -1<x<1$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \; ; \quad -1 < x < 1.$$

Ingretate both sides with respect to x

$$\int \frac{dx}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n \cdot \int x^{2n}\, dx \; ; -1 < x < 1.$$

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1} + C;$$

$-1 < x < 1$

Plug $x = 0$ into both sides.

$\arctan(0) = C$. So, $C = 0$

$$\longrightarrow \boxed{\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}; \quad -1 < x < 1}$$

HW #2. Find the power series centered at 0 for the function $f(x) = \boxed{\dfrac{x^7}{(8+x^8)^2}}$. using differentiation. (start out with a function related to the given function, find a power series for that & differentiate)

$$\frac{1}{8+x^8} = \frac{1}{8\left(1+\frac{x^8}{8}\right)} = \frac{1}{8\left(1-\left(-\frac{x^8}{8}\right)\right)}$$

$$= \frac{1}{8}\cdot\sum_{n=0}^{\infty}\left(-\frac{x^8}{8}\right)^n$$

$$= \frac{1}{8}\sum_{n=0}^{\infty}\frac{(-1)^n}{8^n}\cdot x^{8n}$$

$$\frac{1}{8+x^8} = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n}{8^{n+1}}\cdot x^{8n}}$$

$$\boxed{\left(\frac{1}{u}\right)' = -\frac{u'}{u^2}}$$

$\longrightarrow$ Differentiate both sides.

$$\frac{-8x^7}{(8+x^8)^2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{8^{n+\cancel{1}}}\cdot\cancel{8}n\cdot x^{8n-1}$$

$$\cancel{\left(-\frac{1}{8}\right)}\cdot\frac{\cancel{-}8x^7}{(8+x^8)^2} = \left(-\frac{1}{8}\right)\sum_{n=1}^{\infty}\frac{(-1)^n}{8^n}x^{8n-1}$$

$$\frac{x^7}{(8+x^8)^2} = -\frac{1}{8}\sum_{n=1}^{\infty}\frac{(-1)^n}{8^n}\cdot x^{8n-1}$$

$$= \sum_{n=1}^{\infty}\left(-\frac{1}{8}\right)^{n+1}\cdot x^{8n-1}$$

#7. Find the sum $\sum_{n=1}^{\infty}\frac{n}{2^n}$?

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad ; -1 < x < 1$$

$\longrightarrow$ Differentiate:

$$f'(x) = \sum_{n=1}^{\infty} n\,x^{n-1} = \frac{1}{(1-x)^2} \quad ; -1<x<1$$

Plug $x = \frac{1}{2}$ into the equation:

$$\sum_{n=1}^{\infty} n\cdot\left(\frac{1}{2}\right)^{n-1} = \frac{1}{\left(1-\frac{1}{2}\right)^2} = 4$$

$$\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = 4$$

Multiply both sides by $\frac{1}{2}$:

$$\frac{1}{2} \sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = 2$$

$$\boxed{\sum_{n=1}^{\infty} \frac{n}{2^{n}} = 2}$$