6.3. Taylor Series and MacLaurin Series

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<u>Recall:</u> If a function has the form:

$$\frac{1}{1-(\text{Stuff})}, \text{ then :}$$

$$\frac{1}{1-(\text{Stuff})} = \sum_{n=0}^{\infty} (\text{Stuff})^n ; \quad |\text{Stuff}| < 1$$

Series for the function

We can also differentiate and integrate series for given functions to derive series for other functions.

We want a method to find series for <u>any</u> function.

$\longrightarrow$ Taylor Series & MacLaurin Series.

Taylor Theorem

If f has a power series expansion centered at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} \boxed{c_n}(x-a)^n \; ; \quad |x-a| < R$$

(I.O.C. is $(a-R, \; a+R)$)

$a-R \qquad a \qquad a+R$

Then the coefficients c_n of the series are given by the formula:

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

In other words, the Taylor series for function $f(x)$ is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \cdots$$

In the special case that the center $a = 0$, The Taylor series for $f(x)$ is called the Maclaurin series for $f(x)$, the Maclaurin series is:

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n}.$$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

So, Maclaurin series = Taylor series with center $a = 0$.

E.g. Find the Maclaurin series and its interval of convergence and radius of convergence for the given function.

(a) $f(x) = e^x$ (b) $f(x) = \sin x$ (c) $f(x) = \cos x$

Sol: (a) $f(x) = e^x$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$\longrightarrow$ Find $\dfrac{f^{(n)}(0)}{n!}$ for all $n = 0, 1, 2, \ldots$

$n=0 \longrightarrow c_0 = f(0) \longrightarrow c_0 = e^0 = 1$

$\longrightarrow \boxed{c_0 = 1}$

$n = 1 \longrightarrow c_1 = \dfrac{f'(0)}{1!} \longrightarrow c_1 = \dfrac{1}{1!} = 1.$

$\left(f'(x) = e^x \rightarrow f'(0) = 1 \right)$

$n = 2 \longrightarrow c_2 = \dfrac{f''(0)}{2!} = \dfrac{1}{2!}$

$n = 3 \longrightarrow c_3 = \dfrac{1}{3!}.$

So, in general, $\boxed{c_n = \dfrac{1}{n!}}$ for all $n = 0, 1, 2, 3 \ldots$

$$e^x = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}} \longrightarrow$$ Maclaurin series for e^x.

Interval and Radius of Convergence.

Ratio Test : $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right|$

$$= \frac{|x|}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|}{n+1} = 0 < 1$$

regardless of what x is.

$\longrightarrow$ series converges for all values of x.

$\longrightarrow$ Interval of Convergence: $(-\infty, \infty)$

$R = \infty$

* Important Result:

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \; ; \text{ for } -\infty < x < \infty}$$

Quick application: $e^{1.5}$

$$e^{1.5} = \sum_{n=0}^{\infty} \frac{(1.5)^n}{n!} = \underbrace{1 + 1.5 + \frac{(1.5)^2}{2} + \frac{(1.5)^3}{6} + \cdots}$$

(b) $f(x) = \sin x$.

$$c_n = \frac{f^{(n)}(0)}{n!}.$$

Find $f(0)$; $f'(0)$; $f''(0)$; $f'''(0)$; $f^{(4)}(0)$; etc...

$$\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}$$

Find I.O.C and R.O.C.

<u>Ratio test:</u> $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}\cdot x^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{(-1)^n\cdot x^{2n+1}}\right|$

$$= \frac{|x|^2}{(2n+3)(2n+2)}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|^2}{(2n+3)(2n+2)} = 0 < 1$$

The series converges for all x.

<u>Result:</u> Maclaurin Series for sin x

$$\boxed{\sin x = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n+1}}{(2n+1)!} \quad ; -\infty < x < \infty}$$

(c) $f(x) = \cos x$

$$\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{\boxed{2n}}}{(\boxed{2n})!}$$

$f(x) = \cos x \to f(0) = 1 \to c_0 = 1 \to 1$

$f'(x) = -\sin x \to f'(0) = 0 \to c_1 = 0 \to 0$

$f''(x) = -\cos x \to f''(0) = -1 \to c_2 = \frac{-1}{2!} \to -\frac{x^2}{2!}$

$f'''(x) = \sin x \to f'''(0) = 0 \to c_3 = 0 \to 0$

$f^{(4)}(x) = \cos x \to f^{(4)}(0) = 1 \to c_4 = \frac{1}{4!} \to \frac{x^4}{4!}$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots$$

$2^{\underline{nd}}$ way to get this series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

$$-\infty < x < \infty$$

$\longrightarrow$ Take the derivative of both sides:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$-\infty < x < \infty$$