Result: $$\boxed{\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \; ; \; -\infty < x < \infty}$$

Some important Maclaurin Series:

Function	Maclaurin Series	I.O.C.
$f(x) = e^x$	$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$	$-\infty < x < \infty$ (converges for all x)
$f(x) = \sin x$	$\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$	$-\infty < x < \infty$
$f(x) = \cos x$	$\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$	$-\infty < x < \infty$
$f(x) = \frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$	$-1 < x < 1$
$f(x) = \arctan x$	$\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$	$-1 < x < 1$

Taylor and Maclaurin Polynomials:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots$$

$P_0(x) = 1 \longrightarrow 0^{\underline{th}}$ degree Maclaurin polynomial for e^x

$P_1(x) = 1 + x \longrightarrow 1^{\underline{st}}$ ———————

$P_2(x) = 1 + x + \frac{x^2}{2} \longrightarrow 2^{\underline{nd}}$ degree ———————

$\vdots$

$P_5(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} \longrightarrow 5^{\underline{th}}$ degree.

Approximate $e^{1.5}$: $P_5(1.5) \approx 4.46172$

≈ 4.48169

Definition of the n^{th} degree Taylor polynomial and Maclaurin polynomial of a function:

If $f(x)$ has n derivatives at a, then the n^{th} degree Taylor poly. for f centered a is:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}$$

$$\boxed{P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

n^{th} degree Maclaurin polynomial is

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n.$$

Ex. (1) Find the $\overbrace{4^{th} \text{ degree Maclaurin polynomial}}^{P_4(x)}$ for

$$f(x) = \ln(1+x)$$

(2) Use $P_4(x)$ to approximate $\ln(1.1)$

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(1) $$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$f(0) = \ln(1) = 0$$

$$f'(x) = \frac{1}{1+x} \; ; \; f'(0) = 1$$

$$f''(x) = \frac{-1}{(1+x)^2} \; ; \; f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3} \; ; \; f'''(0) = 2$$

$$f^{(4)}(x) = \frac{-6}{(1+x)^4} \; ; \; f^{(4)}(0) = -6$$

$$P_4(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \longrightarrow f(x) = \ln(1+x)$$

② To approximate $\ln(1.1) = \ln(1+0.1)$ by

$$P_4(0.1) = 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4}$$

$$= 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4}$$

$$P_4(0.1) \simeq 0.09531$$

$$\ln(1.1) = 0.09531018\ldots$$

Taylor Remainder Theorem

Assume f is differentiable $(n+1)$ times on an interval I containing a.

$P_n(x) :=$ $n^{\underline{th}}$ degree Taylor polynomial centered at a.

let $\underbrace{R_n(x)}_{n^{\text{th}}\text{ Taylor Remainder}} = f(x) - P_n(x)$

The followings hold :

(1) The Taylor series for f converges to f at x if and only if $\lim_{n\to\infty} R_n(x) = 0$.

(2) Upper bound for the remainder $R_n(x)$:

$$\underbrace{|R_n(x)|}_{\text{error}} \le \underbrace{\frac{M}{(n+1)!}|x-a|^{n+1}}_{\text{upper bound for error}}$$

where M is a number such that $|f^{(n+1)}(x)| \le M$ for all x in I.

E.g. $f(x) = \sqrt{x}$

① Find the 1^{st} and 2^{nd} degree Taylor polynomials for f centered at $\boxed{a = 4}$. ($p_1(x)$; $p_2(x)$)

② Use $p_1(x)$ and $p_2(x)$ to approximate $\sqrt{6} = f(6)$

Find upper bounds for $R_1(6)$ and $R_2(6)$.

$p_1(x) = f(a) + f'(a)(x-a)$. Here $a = 4$

$f(4) = \sqrt{4} = 2$. $f'(x) = \frac{1}{2\sqrt{x}}$; $f'(4) = \frac{1}{4}$

$\frac{1}{2}x^{-1/2}$

$$\boxed{p_1(x) = 2 + \frac{1}{4}(x-4)}$$

$p_2(x) = \underbrace{f(a) + f'(a)(x-a)} + \underbrace{\frac{f''(a)}{2}(x-a)^2}$

$f''(x) = -\frac{1}{4x^{3/2}} \rightarrow f''(4) = -\frac{1}{4\cdot(4)^{3/2}} = -\frac{1}{32}$

$$\boxed{p_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2}$$

1st approx.

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2nd approx.

(2) $P_1(6) = 2 + \frac{1}{4}(6-4) = \boxed{2.5}$

$P_2(6) = 2 + \frac{1}{4}(6-4) - \frac{1}{64}(6-4)^2 = \boxed{2.4375}$

According Taylor Remainder Theorem.

$|R_1(x)| \leq \frac{M}{2!}(x-a)^2$; M is an upper bound for $|f^{(2)}(x)|$

Here: $x = 6$; $a = 4$; M ?

$|f''(x)| = \left|\frac{-1}{4x^{3/2}}\right|$ on the interval $(4,6)$

$$M = \frac{1}{4(4)^{3/2}} = \frac{1}{32}.$$

$$|R_1(x)| \leq \frac{1/32}{2} \cdot (6-4)^2 = \frac{1}{32} \cdot 2 = \frac{1}{16}.$$

$= 0.0625$

Error when using $p_1(x)$ to estimate $f(x)$ at $x = 6$ is no more than 0.0625