6.4. Working with Taylor Series and Maclaurin S

Thursday, April 12, 2018 1:02 PM

Function	Maclaurin Series	I.O.C
$f(x) = \frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$	$-1 < x < 1$
$f(x) = e^x$	$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$	$-\infty < x < \infty$
$f(x) = \sin x$	$\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$-\infty < x < \infty$
$f(x) = \cos x$	$\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$-\infty < x < \infty$
$f(x) = \arctan(x)$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} =$ $x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$	$-1 < x < 1$

* Find the MacLaurin series for $f(x) = \ln(1+x)$.

$$\ln(1+x) = \int \boxed{\frac{1}{1+x}}\, dx$$

this has a nice series representation

$$\frac{1}{1+x} = \frac{1}{1-\boxed{(-x)}} = \boxed{\sum_{n=0}^{\infty} (-x)^n}$$

Stuff

; converges when

$$|-x| < 1$$
$$\leftrightarrow |x| < 1$$
$$\leftrightarrow -1 < x < 1$$

$$\ln(1+x) = \int \sum_{n=0}^{\infty} (-x)^n\, dx = \sum_{n=0}^{\infty} (-1)^n \int x^n\, dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{n+1}}{n+1} + C.$$

Plug $x = 0$ into both the function and the series:

$$\ln(1) = C \longrightarrow C = 0$$

$$\boxed{\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{n+1}}{n+1} \; ; \quad -1 < x < 1}$$

→ Find Maclaurin series for functions of the form

$f(x) = (1+x)^{\boxed{r}}$ where r is a real number

E.g. $f(x) = (1+x)^{1/2} = \sqrt{1+x}$

$f(x) = (1+x)^{3/7} = \sqrt[7]{(1+x)^3}$

The series for these functions are called the binomial series.

binomial coefficients

Binomial Series Formula:

$$\boxed{(1+x)^R = \sum_{n=0}^{\infty} \boxed{\binom{R}{n}} x^n \; ; \text{ converges for } -1 < x < 1}$$

Here $\binom{R}{n}$ (read as R choose n) are the binomial coefficients whose formula is:

$$\binom{R}{n} = \frac{R(R-1)(R-2)\cdots(R-n+1)}{n!}$$

coeff. of series

So, we have:

$$(1+x)^R = \sum_{n=0}^{\infty} \boxed{\frac{R(R-1)(R-2)\cdots(R-n+1)}{n!}} \cdot x^n \quad ; -1 < x < 1$$

$$= 1 + R\cdot x + \frac{R(R-1)}{2!}x^2 + \frac{R(R-1)(R-2)}{3!}x^3 + \frac{R(R-1)(R-2)(R-3)}{4!}x^4 + \cdots$$

E.g. $f(x) = \sqrt{1+x} = (1+x)^{1/2}$.

(1) Find the binomial series for f. Write down the series (with coeffs.) up to x^4.

(2) Find the 3^{rd} degree Maclaurin polynomial $P_3(x)$ for f. Use it to estimate $f(0.5) = \sqrt{1.5}$.

③ Use the Taylor's Remainder Theorem from last time to find an upper bound for the error.

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① $\to R = 1/2$

$$(1+x)^{1/2} = 1 + \frac{x}{2} + \boxed{\frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)}{2!}}x^2 + \boxed{\frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}}x^3 + \boxed{\frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)\cdot\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{4!}}x^4 + \dots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots$$

$$(1+x)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n$$

② $$P_3(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}.$$

$$P_3(0.5) = 1 + \frac{0.5}{2} - \frac{(0.5)^2}{8} + \frac{(0.5)^3}{16} \approx \boxed{1.2266}$$

$\to$ approx. for $f(0.5) = \sqrt{1.5}$

3) $R_3(x) = f(x) - P_3(x)$

Taylor's Remainder Theorem

$$|R_3(x)| \le \frac{M}{4!}|x-a|^4 \; ; \quad a = 0$$

M is an upper bound for $f^{(4)}(x)$

on $I = (0, 0.5)$

$\longrightarrow$ find $f^{(4)}(x)$.

$f(x) = (1+x)^{1/2}$; $f'(x) = \frac{1}{2}(1+x)^{-1/2}$

$f''(x) = -\frac{1}{4}(1+x)^{-3/2}$; $f'''(x) = \frac{3}{8}(1+x)^{-5/2}$

$$|f^{(4)}(x)| = \left|-\frac{15}{16}\cdot(1+x)^{-7/2}\right| = \left|-\frac{15}{16(1+x)^{7/2}}\right|$$

$\longrightarrow$ upper bound of this on $(\boxed{0}, 0.5)$

$$\boxed{M = \frac{15}{16}} \longrightarrow |R_3(x)| \le \underbrace{\frac{15/16}{4!}\cdot(0.5)^4}_{\approx 0.00244}$$

* Derive Maclaurin Series for new functions using known series.

E.g. Find the Maclaurin series for the function:

$$f(x) = e^{x+3}.$$

Find I.O.C.

We already know: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$; $-\infty < x < \infty$

Rewrite $f(x)$ as: $f(x) = e^{x+3} = \boxed{e^3} \cdot \boxed{e^x}$

constant

Thus, $f(x) = e^{x+3} = e^3 \cdot \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{e^3}{n!} \cdot x^n$

coeffs. for the series.

I.O.C: $-\infty < x < \infty$