Thursday, April 12, 2018 1:58 PM

E.g. Find the Maclaurin series for

$$g(x) = \cos(\sqrt{x}).$$

Find I.O.C.

We know: $\cos(u) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{u^{2n}}{(2n)!}$; $-\infty < u < \infty$

So, $\cos(\sqrt{x}) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(\sqrt{x})^{2n}}{(2n)!}$ $\longrightarrow$ $\left[(\sqrt{x})^2\right]^n$

$$\cos(\sqrt{x}) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^n}{(2n)!}$$

I.O.C: $[0, \infty)$

* Find the Maclaurin series for the given function.

① $f(x) = e^{x/3}$ ② $g(x) = (1-8x)^{5/6}$

③ $h(x) = \dfrac{\sin(\sqrt{x})}{\sqrt{x}}$ ④ $j(x) = \sinh(x)$

$$\underbrace{\sinh(x)}_{\text{sine hyperbolic}} = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

⑤ $k(x) = \sin^2(x)$.

$\left(\text{Hint: } \sin^2(x) = \dfrac{1}{2} - \dfrac{1}{2}\cos(2x)\right)$

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① $f(x) = e^{x/3}$

$$e^{u} = \sum_{n=0}^{\infty} \frac{u^n}{n!} \quad ; \quad -\infty < u < \infty$$

$u = \frac{x}{3}$

$$\longrightarrow e^{\frac{x}{3}} = \sum_{n=0}^{\infty} \frac{\left(\frac{x}{3}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{\frac{x^n}{3^n}}{n!}$$

$$e^{\frac{x}{3}} = \sum_{n=0}^{\infty} \frac{x^n}{3^n n!} \quad ; \quad -\infty < x < \infty$$

② $g(x) = (1 - 8x)^{5/6}$

$$(1+u)^R = \sum_{n=0}^{\infty} \binom{R}{n} u^n .$$

$$g(x) = (1 + (-8x))^{5/6}$$

$$= \sum_{n=0}^{\infty} \binom{5/6}{n} \cdot (-8x)^n$$

$$= \sum_{n=0}^{\infty} \binom{5/6}{n} \cdot (-8)^n \cdot x^n$$

③ $h(x) = \dfrac{\sin(\sqrt{x})}{\sqrt{x}} = \dfrac{1}{\sqrt{x}} \cdot \sin(\sqrt{x})$

$$= \frac{1}{\sqrt{x}} \cdot \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(\sqrt{x})^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(\sqrt{x})^{2n+\cancel{1}}}{\cancel{\sqrt{x}} \cdot (2n+1)!}$$

$$\frac{\sin(\sqrt{x})}{\sqrt{x}} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^n}{(2n+1)!}$$

④ $j(x) = \frac{1}{2}\cdot\left(e^{x} - e^{-x}\right)$

$$= \frac{1}{2}\cdot\left(\sum_{n=0}^{\infty}\frac{x^n}{n!} - \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\right)$$

$$= \frac{1}{2}\cdot\left(\sum_{n=0}^{\infty}\frac{x^n}{n!} - \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}\right)$$

$$= \frac{1}{2}\cdot\left(\sum_{n=0}^{\infty}\underbrace{\left(1-(-1)^n\right)}_{a}\frac{x^n}{n!}\right)$$

* If n is even, $1-(-1)^n = 1-1 = 0$

→ all the even powers of x go away.

* If n is odd, $1-(-1)^n = 1-(-1) = 2$

→ the odd powers of x survive.

$$= \sum_{m=0}^{\infty}\frac{x^{2m+1}}{(2m+1)!}$$

Thursday, April 12, 2018 2:52 PM

5) $\sin^2 x = \frac{1}{2} - \frac{1}{2}\boxed{\cos(2x)}$

$$\cos(u) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{u^{2n}}{(2n)!} \; ; \quad u = 2x$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2} \cdot \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(2x)^{2n}}{(2n)!}$$

$$\sin^2 x = -\frac{1}{2}\sum_{n=1}^{\infty} (-1)^n \cdot \frac{4^n \cdot x^{2n}}{(2n)!}$$

$$= -\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 4^n}{2 \cdot (2n)!} \cdot x^{2n} .$$

Ex. Find the 25^{th} derivative of

$f(x) = (1 + x^2)^{15}$ at $x = 0$

(find $f^{(25)}(0)$)

Apply the Binomial Series:

$$(1+x^2)^{15} = \sum_{n=0}^{\infty} \binom{15}{n} (x^2)^n$$

$$= \sum_{n=0}^{\infty} \binom{15}{n} x^{2n}$$

So, the 25$^{\underline{th}}$ derivative is included in the 25$^{\underline{th}}$ coefficient of this series.

$$\frac{f^{(25)}(0)}{25!} = c_{25}$$

coeff. of x^{25}

$$\frac{f^{(25)}(0)}{25!} = 0 \rightarrow f^{(25)}(0) = \boxed{0}.$$

Ex.

$$F(x) = \int_0^x \boxed{\frac{\ln(1+t)}{t}}.$$

Find the Maclaurin series for $F(x)$.

$$\ln(1+t) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{t^{n+1}}{n+1}$$

$$\frac{\ln(1+t)}{t} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{t^n}{n+1}$$

$$\int_0^x \frac{\ln(1+t)}{t}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \int_0^x t^n\,dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \cdot \left.\frac{t^{n+1}}{n+1}\right|_0^x$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} \cdot x^{n+1}$$