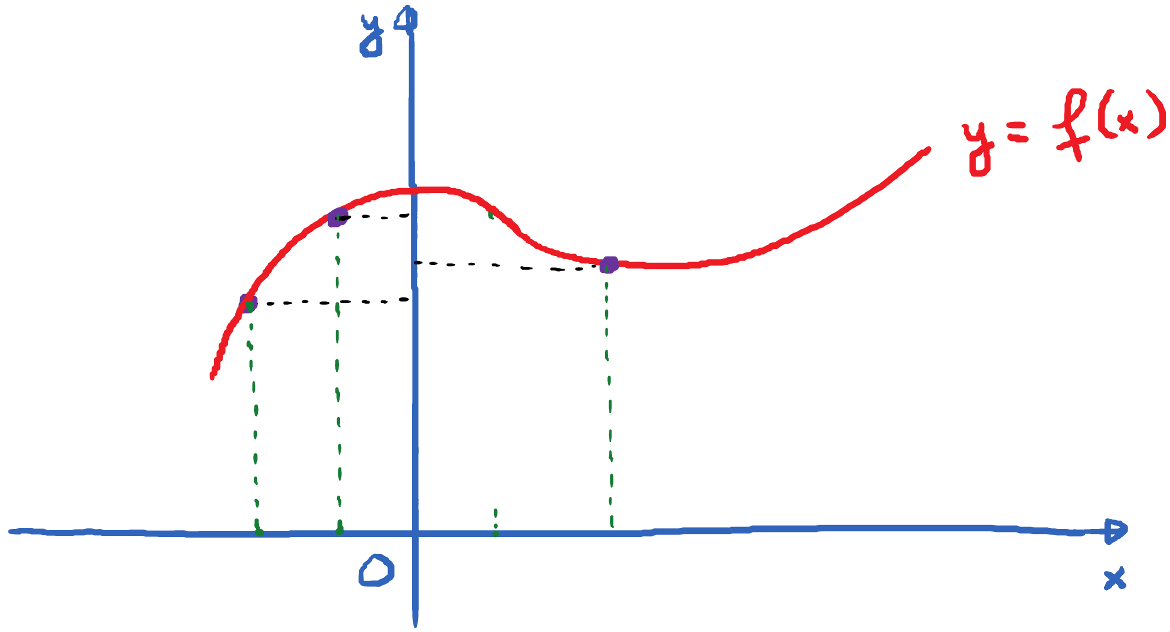
7.1 and 7.2. Parametric Curves and Calculus of Parametric Curves

Tuesday, April 17, 2018 1:03 PM

What is a parametric curve?



x and y coordinates of a point moving along this curve change with respect to time.

Introduce the variable t for time

→ Both x and y are functions of t

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

These 2 equations are called parametric equations for the curve $y = f(x)$.

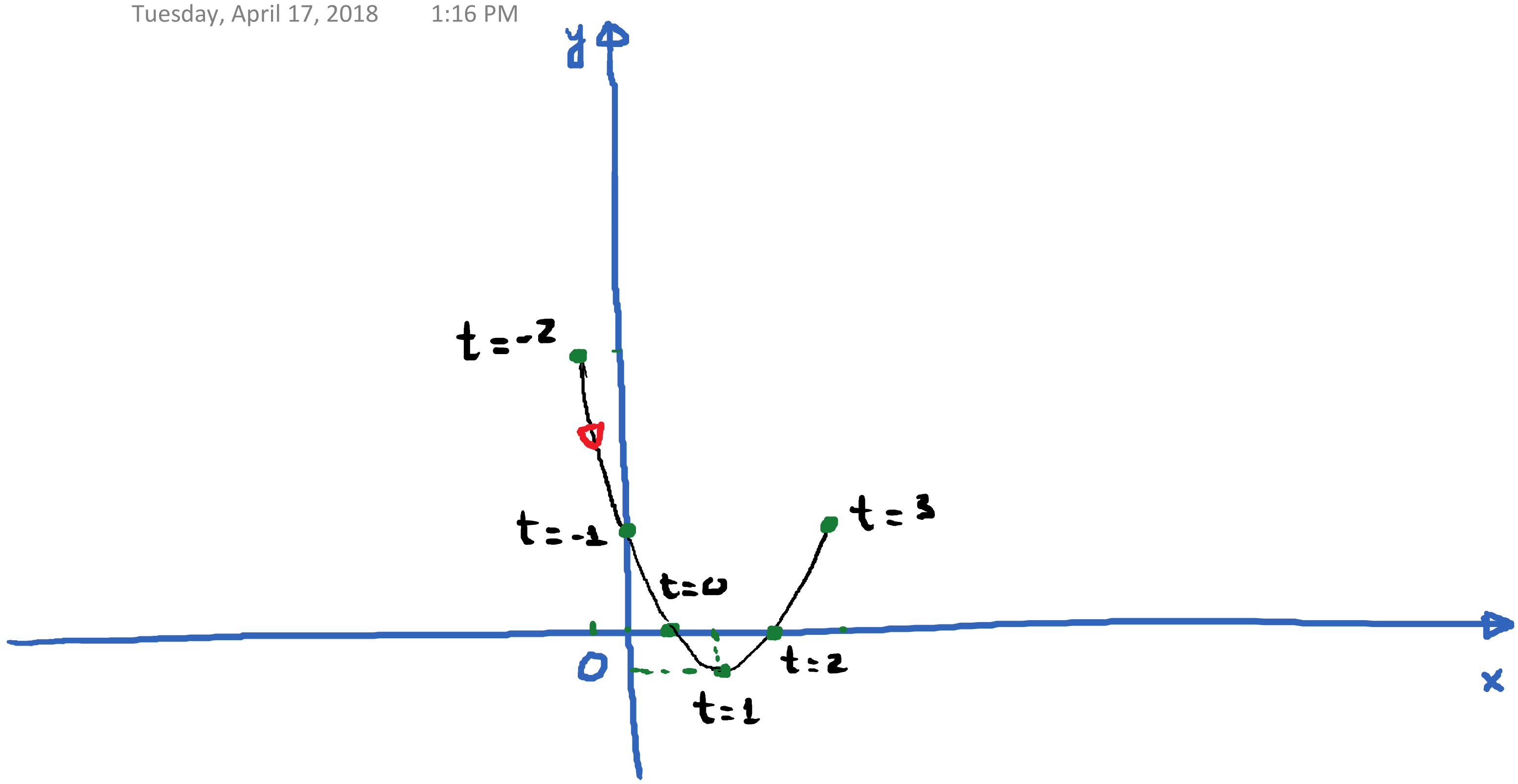
E.g. $\begin{cases} x = x(t) = t + 1 \\ y = y(t) = t^2 - 2t \end{cases}$

$\longrightarrow$ parametric equations for a parametric curve in the xy-plane. (t is called the parameter)

t	$x = t+1$	$y = t^2 - 2t$	(x, y)
-2	-1	8	$(-1, 8)$
-1	0	3	$(0, 3)$
0	1	0	$(1, 0)$
1	2	-1	$(2, -1)$
2	3	0	$(3, 0)$
3	4	3	$(4, 3)$

point on curve at time $t = -2$

pt on curve at time $t = -1$



To know the shape of the graph, in many cases, it is useful to have an equation of y in terms of x, rather than 2 equations of x and y in terms of t.

$\longrightarrow$ Elimination of parameter.

$$\begin{cases} x = x(t) = t + 1 \\ y = y(t) = t^2 - 2t \end{cases}$$

<u>Want</u>: Eliminate t and get an equation that relates x and y.

Idea: Solve for t in one of the equation

→ t in terms of x (or y)

Then plug it back to the other equation.

$$\begin{cases} x = t + 1 \\ y = t^2 - 2t \end{cases} \qquad x = t+1 \longrightarrow t = \boxed{x - 1}$$

$$\longrightarrow y = (x-1)^2 - 2(x-1)$$

$$\longrightarrow y = x^2 - 2x + 1 - 2x + 2$$

$$y = x^2 - 4x + 3.$$

So we know that the curve is a parabola.

Since $a = 1 > 0$, parabola points upward.

Vertex: x-vertex $= -\dfrac{b}{2a} = \dfrac{4}{2} = 2$

y-vertex $= (2)^2 - 4(2) + 3 = -1$

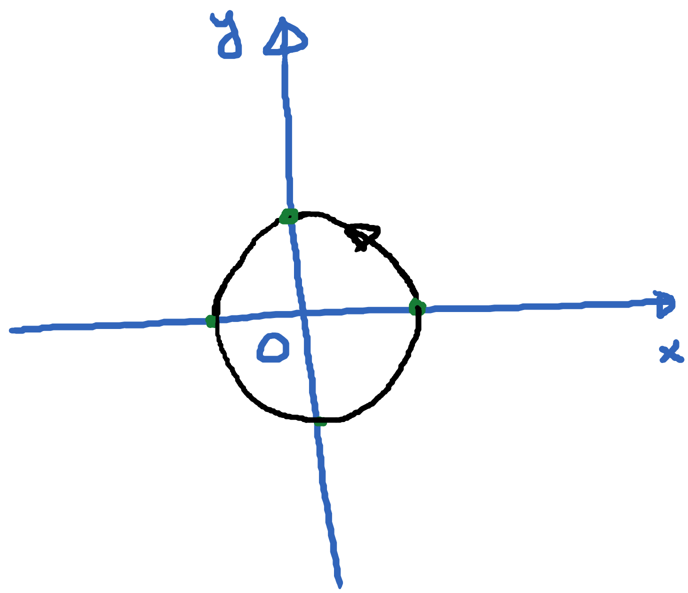
Vertex $= (2, -1)$

E.g. Given the parametric curve

$$\begin{cases} x = x(t) = \cos(t) \\ y = y(t) = \sin(t) \end{cases} ; \quad 0 \leq t \leq 2\pi$$

Q: What curve is this?

t	x	y	(x, y)
0	1	0	$(1,0)$
$\frac{\pi}{2}$	0	1	$(0,1)$
π	-1	0	$(-1,0)$
$\frac{3\pi}{2}$	0	-1	$(0,-1)$
2π	1	0	$(1,0)$



Elimination of parameter:

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

So, $x^2 + y^2 = 1$ $\longrightarrow$ this curve is the unit circle.

Calculus of Parametric Curves

Tangent line Problem:

If curve is given as $y = f(x)$, to find the tangent line to the curve at (a, b), we first find $f'(x)$.

Then Slope of tangent line $= f'(a)$.

Equation of tangent line : $y - b = f'(a)(x - a)$

Now, the curve is given as :

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

Q: How do we find the slope of the tangent line at a point on the curve?

E.g. Given the parametric curve:

$$\begin{cases} x = x(t) = t^2 - 4t \\ y = y(t) = 2t^3 - 6t \end{cases} ; \quad -2 \le t \le 3.$$

Find the equation of the tangent line to the graph of this curve at the point where $\boxed{t = 1.}$

When $t = 1$: $x = -3$; $y = -4$

$\longrightarrow$ point $(-3, -4)$.

Need: Slope = ?

$$\text{Slope} = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)} = \frac{6t^2 - 6}{2t - 4}$$

$$\text{Slope at } (-3;-4) = \frac{6\cdot(1)^2 - 6}{2\cdot(1) - 4} = \frac{0}{-2} = 0$$

Equation of tangent line at $(-3,-4)$: $\boxed{y = -4}$