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In general, given a curve defined via 2 parametric equations:

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

Suppose $x'(t)$ and $y'(t)$ exist and $x'(t) \neq 0$.

Then $$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}}$$

Suppose $y'(t) \neq 0$

Then $$\boxed{\frac{dx}{dy} = \frac{dx/dt}{dy/dt} = \frac{x'(t)}{y'(t)}}$$

Ex. Given the parametric curve

$$x = t^5 - 4t^3 \; ; \quad y = t^2$$

Q: Find the tangent line to the curve when $t = 2$.

Ans: $y = \frac{1}{8}x + 4$

Ex. Given the parametric curve in the xy-plane:

$$x = t^3 - 3t \;;\; y = 3t^2 - 9.$$

Q: Find the x-y-coordinates of the point(s) at which the curve have horizontal tangent line or vertical tangent line.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \boxed{\frac{y'(t)}{x'(t)}}$$

Horizontal tangent line: $y'(t) = 0 \;;\; x'(t) \neq 0$

Vertical tangent line: $x'(t) = 0 \;;\; y'(t) \neq 0.$

$x'(t) = 3t^2 - 3. \quad y'(t) = 6t$

$\longrightarrow$ Horizontal tangent line: $\begin{cases} 6t = 0 \\ 3t^2 - 3 \neq 0 \end{cases} \longrightarrow \boxed{t = 0}$

$\longrightarrow x = 0 \;;\; y = -9 \longrightarrow \boxed{(0,-9)}.$

$\longrightarrow$ Vertical tangent line: $\begin{cases} 3t^2 - 3 = 0 \\ 6t \neq 0 \end{cases} \longrightarrow t = \pm 1$

$t = 1 \rightarrow \boxed{(-2,-6)} \;;\quad t = -1 \rightarrow \boxed{(2,-6)}$

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* How to find the second derivative of a parametric curve.

E.g. Parametric curve:

$$\begin{cases} x = x(t) = t^2 - 3 \\ y = y(t) = 2t - 1 \end{cases}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)} = \frac{2}{2t} = \boxed{\frac{1}{t}}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d\left(\frac{dy}{dx}\right)/dt}{dx/dt}$$

operation: take derivative w.r.t. x

$$= \frac{\boxed{\frac{d}{dt}\left(\frac{dy}{dx}\right)}}{\boxed{\frac{dx}{dt}}}$$

→ this tells us to take the derivative w.r.t. the variable t of the expression $\frac{dy}{dx}$

→ this is just $x'(t)$

$$\frac{dy}{dx} = \boxed{\frac{1}{t}}$$

$$(t^{-1})' = -1 \cdot t^{-2} = -\frac{1}{t^2}$$

$$\longrightarrow \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{1}{t}\right)}{x'(t)} = \frac{-\frac{1}{t^2}}{2t} = \boxed{-\frac{1}{2t^3}}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}}$$

Step 1: Find $\frac{dy}{dx}$

Step 2: Find derivative w.r.t. t of $\frac{dy}{dx}$

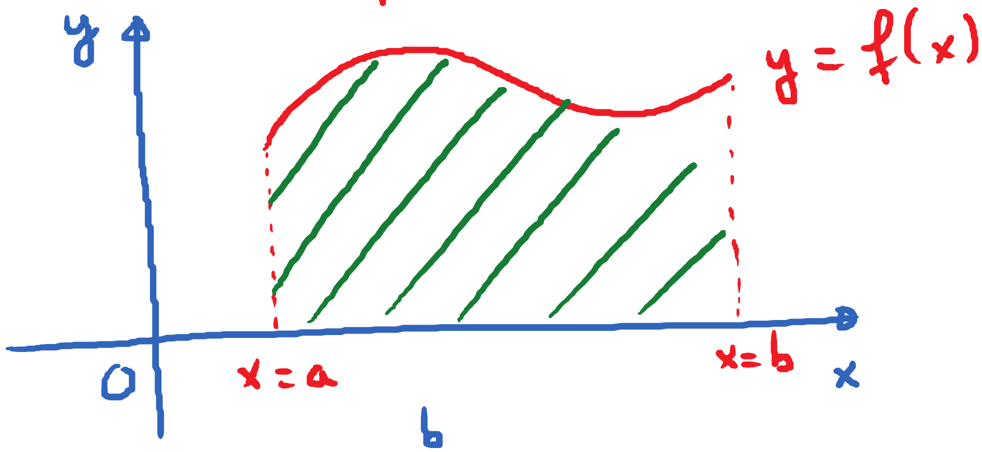
Step 3: Divide the above by $\frac{dx}{dt}$.

Ex. Given the parametric curve:

$$\begin{cases} x = x(t) = t^5 - 4t^3 \\ y = y(t) = t^2 \end{cases}$$

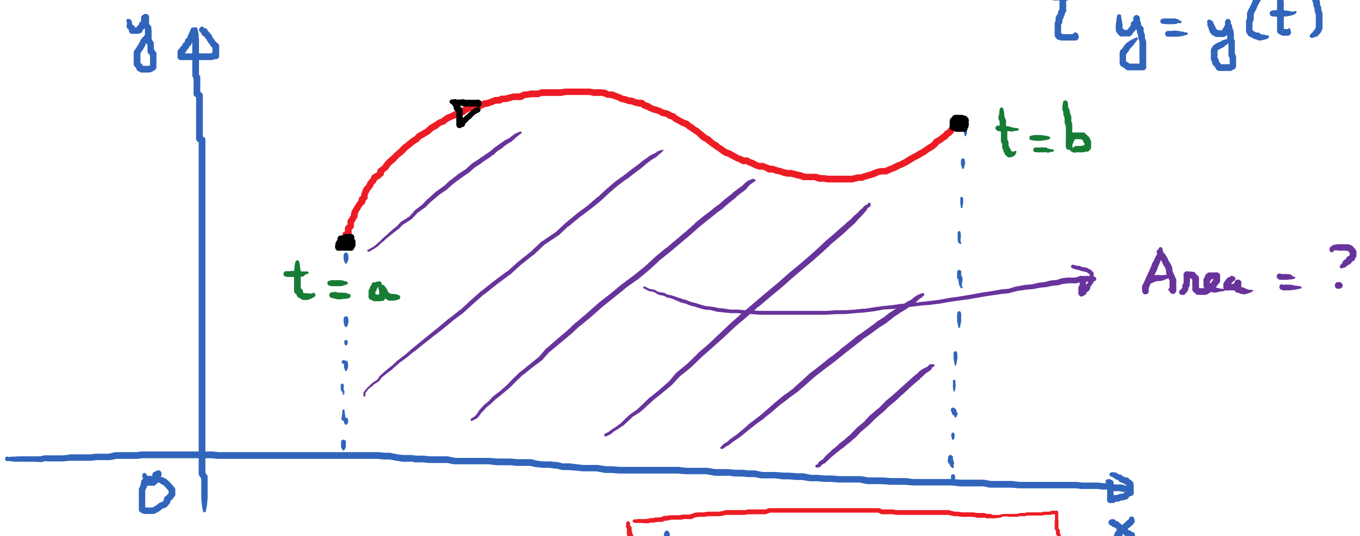
$\longrightarrow$ Find $\frac{d^2y}{dx^2}$? Ans: $\frac{-30t^2+24}{t^4(5t^2-12)^3}$

Area under a parametric curve



Shaded Area $= \int_a^b f(x)\,dx$

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$



$$\text{Area} = \int y\,\boxed{dx} = \boxed{\int_a^b y(t)\cdot x'(t)\,dt}$$

$$\frac{dx}{dt} = x'(t) \longrightarrow dx = \boxed{x'(t)\,dt}$$

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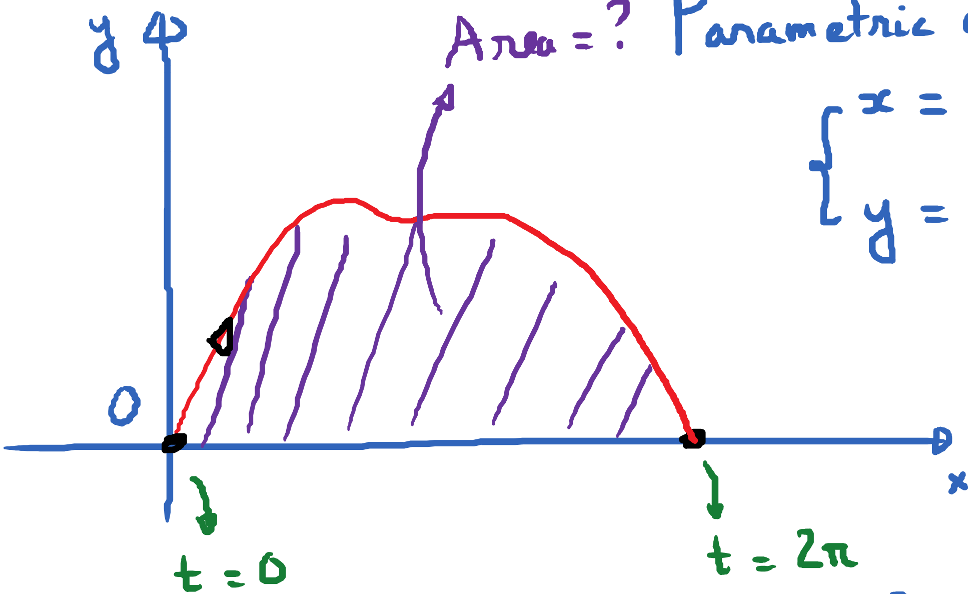


Parametric curve has equations:

$$\begin{cases} x = x(t) = t - \sin(t) \\ y = y(t) = 1 - \cos(t) \end{cases}$$

$$0 \leq t \leq 2\pi$$

$$\text{Area} = \int_0^{2\pi} y(t) \cdot x'(t)\, dt = \int_0^{2\pi} (1-\cos t)\cdot(1-\cos t)\, dt$$

$$= \int_0^{2\pi} (1-\cos t)^2\, dt = \int_0^{2\pi} \left(1 - 2\cos t + \boxed{\cos^2 t}\right) dt$$

Power reduction : $\cos^2 t = \boxed{\dfrac{1+\cos(2t)}{2}}$

$$\int_0^{2\pi} \left(1 - 2\cos t + \frac{1}{2} + \frac{1}{2}\cos(2t)\right) dt$$

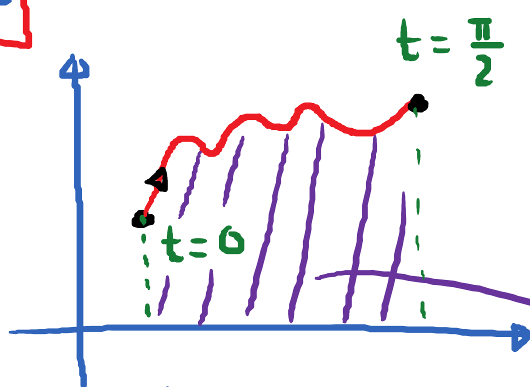
$$= \int_0^{2\pi} \left(\frac{3}{2} - 2\cos t + \frac{1}{2}\cos(2t)\right) dt$$

$$= \left[\frac{3}{2}t - 2\sin t + \frac{1}{4}\sin(2t)\right]\Bigg|_0^{2\pi}$$

$$= \left[\frac{3}{2}(2\pi) - \cancel{2\sin(2\pi)} + \frac{1}{4}\cdot\cancel{\sin(4\pi)}\right] - 0$$

$$= \boxed{3\pi}$$

Ex.



$$\begin{cases} x = x(t) = \cos(t) \\ y = y(t) = e^t \end{cases}$$

$$0 \le t \le \frac{\pi}{2}$$

Area = ?

$$\text{Area} = \int_0^{\pi/2} e^t\cdot(-\sin t)\,dt = -\int_0^{\pi/2} e^t \sin t\,dt$$

Integration by Parts.

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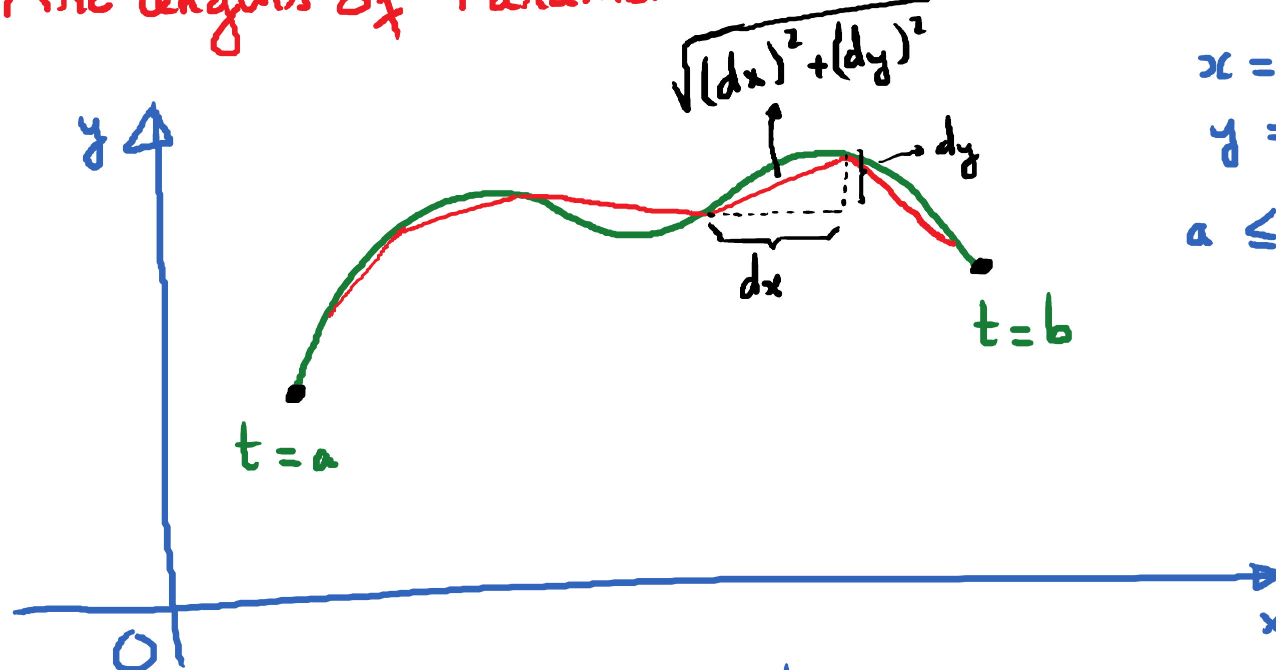
Arc lengths of Parametric Curves



$x = x(t)$

$y = y(t)$

$a \le t \le b$

Length of this curve = ? $\displaystyle\int_a^b \sqrt{(dx)^2 + (dy)^2}$

$$\int_a^b \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right](dt)^2}$$

Formula:

$$\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt}$$