E.g. Find the length of the parametric curve defined by

$$\begin{cases} x = x(t) = 3\cos(t) \\ y = y(t) = 3\sin(t) \end{cases}$$

$$0 \leq t \leq \pi.$$

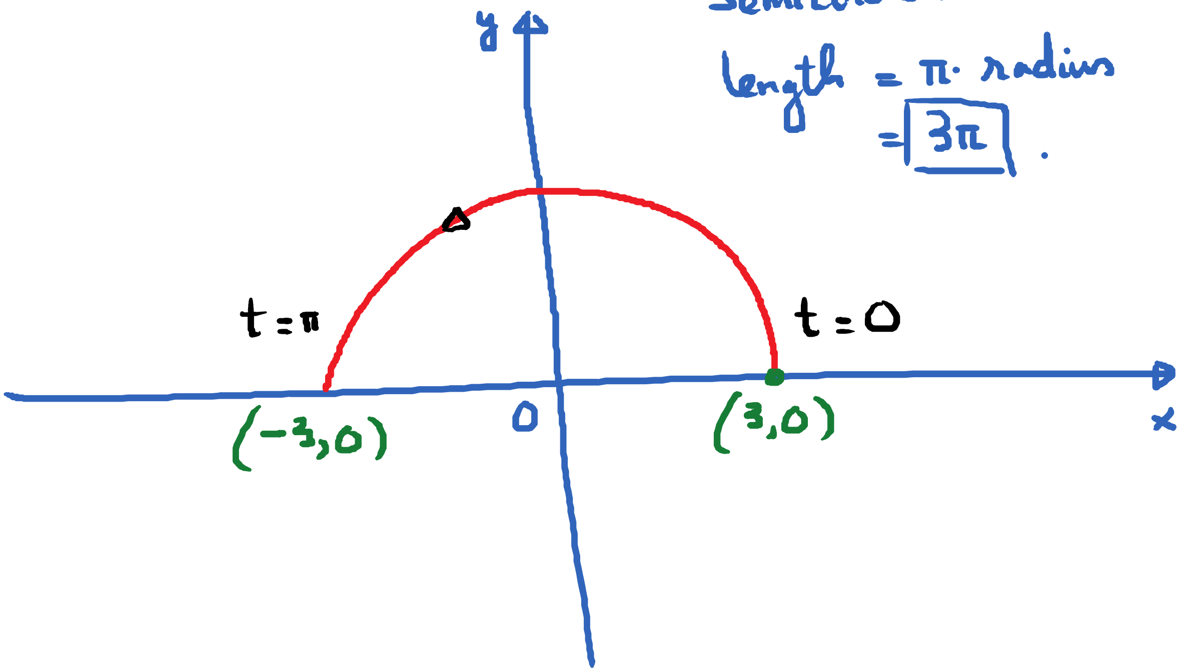
Sol: $x'(t) = -3\sin(t)$; $y'(t) = 3\cos(t)$.

$$L = \int_0^\pi \sqrt{(-3\sin(t))^2 + (3\cos(t))^2}\, dt$$

$$= \int_0^\pi \sqrt{9\sin^2(t) + 9\cos^2(t)}\, dt$$

$$= \int_0^\pi \sqrt{9\underbrace{(\sin^2(t) + \cos^2(t))}_{1}}\, dt$$

$$= \int_0^\pi 3\,dt = 3t\Big|_0^\pi = 3\cdot\pi - 3\cdot 0 = \boxed{3\pi}$$



Semicircle.

length = π · radius
$= \boxed{3\pi}$.

Ex. Find the length of the curve defined by:

$$\begin{cases} x = x(t) = 3t^2 \\ y = y(t) = 2t^3 \end{cases} \quad ; \quad 1 \leq t \leq 3.$$

Sol: $x'(t) = 6t$; $y'(t) = 6t^2$

$$L = \int_1^3 \sqrt{(6t)^2 + (6t^2)^2}\, dt = \int_1^3 \sqrt{36t^2 + 36t^4}\, dt$$

$$= \int_1^3 \sqrt{36t^2(1+t^2)}\, dt = \frac{6}{2}\int_1^3 2t\sqrt{1+t^2}\, dt$$

Let $u = 1 + t^2$. $du = 2t\,dt$. $t = 1 \rightarrow u = 2$

$t = 3 \rightarrow u = 10$

$$\rightarrow \quad 3\int_2^{10} \sqrt{u}\,du = 3 \cdot \int_2^{10} (u)^{1/2}\,du$$

$$= \cancel{3} \cdot \frac{2u^{3/2}}{\cancel{3}} \Bigg|_2^{10}$$

$$= 2\left[(10)^{3/2} - (2)^{3/2}\right]$$

$$= \boxed{2 \cdot \left[10\sqrt{10} - 2\sqrt{2}\right]}$$

Ex. Find the length of the cycloid defined as:

$$x = 5(t - \sin t) \qquad ; \; 0 \leq t \leq 2\pi .$$
$$y = 5(1 - \cos t)$$

Sol: $x'(t) = 5(1 - \cos t)$; $y'(t) = 5 \cdot \sin(t)$

$$[x'(t)]^2 + [y'(t)]^2 = 25(1 - \cos t)^2 + 25 \cdot \sin^2(t)$$

$$= 25\left[(1-\cos t)^2 + \sin^2 t\right]$$

$$= 25\left[1 - 2\cos t + \underbrace{\cos^2 t + \sin^2 t}_{1}\right]$$

$$= 25\left[2 - 2\cos t\right] = 50(1-\cos t)$$

$$L = \int_0^{2\pi} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = \int_0^{2\pi} \sqrt{50(1-\cos t)}\, dt$$

Half-angle formula: $\dfrac{1-\cos(\theta)}{2} = \sin^2\left(\dfrac{\theta}{2}\right)$

$$\int_0^{2\pi} \sqrt{50 \cdot 2 \cdot \sin^2\left(\frac{t}{2}\right)}\, dt = \int_0^{2\pi} \sqrt{100 \sin^2\left(\frac{t}{2}\right)}\, dt$$

$$= \int_0^{2\pi} 10 \sin\left(\frac{t}{2}\right) dt = 10 \cdot 2 \int_0^{2\pi} \sin\left(\frac{t}{2}\right) \overbrace{\frac{1}{2}\, dt}^{du}$$

let $u = \dfrac{t}{2}$; $du = \dfrac{1}{2}\, dt$ $\rightarrow$ $20 \displaystyle\int_0^{\pi} \sin(u)\, du$

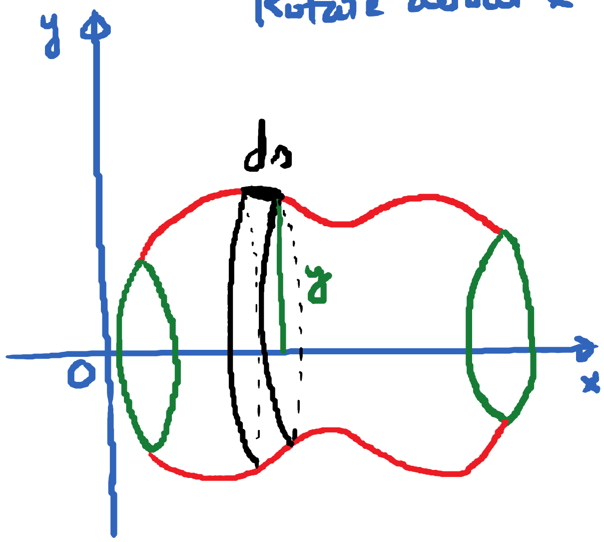
$$20\int_0^{\pi} \sin(u)\,du = -20\cos(u)\Big|_0^{\pi}$$

$$= -20\left[\cos(\pi) - \cos(0)\right]$$

$$= -20\cdot\left[-1-1\right] = \boxed{40}$$

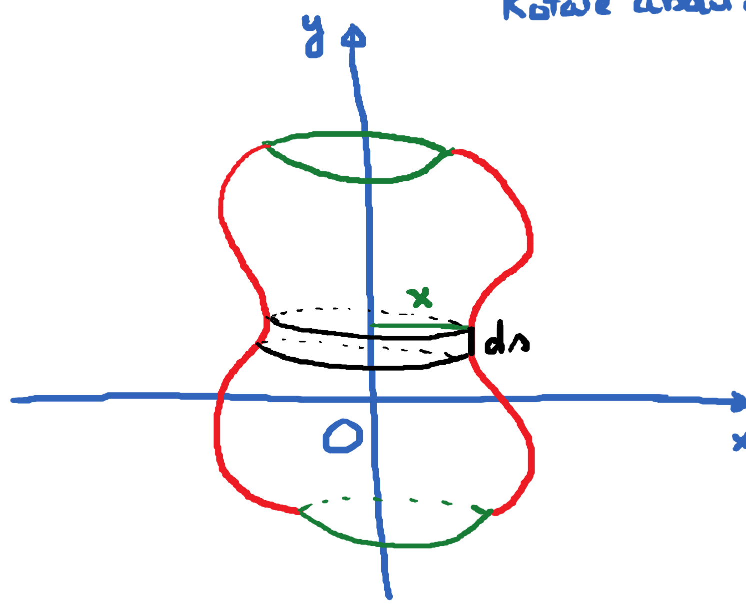
Surface Areas with Parametric Curves

Rotate about x



Surface Area = $\int 2\pi y\, ds$

Rotate about y



Surface area = $\int 2\pi x\, ds$

Now, the red curve is given by parametric equations:

$$x = x(t)\ ;\ y = y(t)\ ;\ a \le t \le b$$

$\longrightarrow$ <u>Formulas</u>:

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

Rotate about x-axis :

$$\boxed{S = \int_a^b 2\pi \cdot y(t) \cdot \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt}$$

Rotate about y-axis :

$$\boxed{S = \int_a^b 2\pi \cdot x(t) \cdot \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt}$$

E.g. Rotate the curve defined by: (R is a constant)

$x = R\cos t$; $y = R\sin t$; $0 \leq t \leq \pi$

about the x-axis. Find the surface area of the resulting surface.

$x'(t) = -R\sin t \;;\quad y'(t) = R\cos t$

$$\text{Surface Area} = \int_0^{\pi} 2\pi \cdot R\sin t \cdot \sqrt{(-R\sin t)^2 + (R\cos t)^2}\, dt$$

$$= \int_0^{\pi} 2\pi \cdot R\sin t \cdot \sqrt{R^2\sin^2 t + R^2\cos^2 t}\, dt$$

$$= \int_0^{\pi} 2\pi \cdot R\sin t \underbrace{\sqrt{R^2(\underbrace{\sin^2 t + \cos^2 t}_{1})}}_{R}\, dt$$

$$= 2\pi R^2 \cdot \int_0^{\pi} \sin t\, dt = 2\pi R^2 \cdot (-\cos t)\Big|_0^{\pi}$$

$$= 2\pi R^2 \left(-\cos\pi + \cos(0)\right) = 2\pi R^2 \cdot (1 + 1) = \boxed{4\pi R^2}.$$

Ex. Find the surface area of the surface obtained by rotating: $x = \cos^3(t)$; $y = \sin^3(t)$; $0 \le t \le \frac{\pi}{2}$ about the x-axis. Ans: $\frac{6\pi}{5}$.