E.g. Consider the polar curve given by the equation

$$\boxed{R = 1 + \sin\theta.}$$

Find the slope of the tangent line to the curve at the point where $\theta = \frac{\pi}{3}$. Use this to obtain the equation of the tangent line at the point.

Sol: $\text{Slope} = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$.

$$\frac{dy}{d\theta} = \frac{dR}{d\theta}\sin\theta + R\cdot\cos\theta$$

$$= \cos\theta\cdot\sin\theta + (1+\sin\theta)\cdot\cos\theta$$

$$\frac{dx}{d\theta} = \frac{dR}{d\theta}\cdot\cos\theta - R\sin\theta$$

$$= \cos\theta\cdot\cos\theta - (1+\sin\theta)\cdot\sin\theta.$$

When $\theta = \frac{\pi}{3}$; $\frac{dy}{d\theta} = \cos\left(\frac{\pi}{3}\right) \cdot \sin\left(\frac{\pi}{3}\right) + \left(1 + \sin\left(\frac{\pi}{3}\right)\right) \cdot \cos\left(\frac{\pi}{3}\right)$

$$= \frac{1}{2} \cdot \frac{\sqrt{3}}{2} + \left(1 + \frac{\sqrt{3}}{2}\right) \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3}}{4} + \frac{1}{2} + \frac{\sqrt{3}}{4} = \frac{1+\sqrt{3}}{2}.$$

$$\frac{dx}{d\theta} = \cos\left(\frac{\pi}{3}\right) \cdot \cos\left(\frac{\pi}{3}\right) - \left(1 + \sin\left(\frac{\pi}{3}\right)\right) \cdot \sin\left(\frac{\pi}{3}\right)$$

$$= \frac{1}{4} - \left(1 + \frac{\sqrt{3}}{2}\right) \cdot \frac{\sqrt{3}}{2} = \frac{1}{4} - \frac{\sqrt{3}}{2} - \frac{3}{4} = \frac{-1-\sqrt{3}}{2}$$

Slope $= \dfrac{\frac{1+\sqrt{3}}{2}}{\frac{-1-\sqrt{3}}{2}} = \boxed{-1}$

Point: (x, y) : $x = R\cos\theta = (1 + \sin\theta) \cdot \cos\theta$

$$x(\pi/3) = \left[1 + \sin\left(\frac{\pi}{3}\right)\right] \cos\left(\frac{\pi}{3}\right)$$

$$= \left(1 + \frac{\sqrt{3}}{2}\right) \cdot \frac{1}{2} = \frac{2+\sqrt{3}}{4}.$$

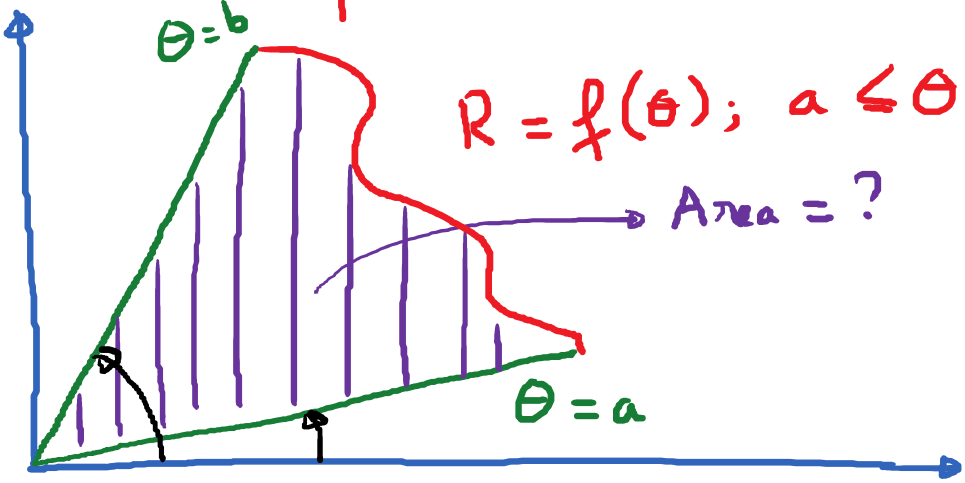
$y = R\sin\theta = (1 + \sin\theta) \cdot \sin\theta$

$$= \left(1 + \sin\left(\frac{\pi}{3}\right)\right) \cdot \sin\left(\frac{\pi}{3}\right) = \left(1 + \frac{\sqrt{3}}{2}\right) \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{3}}{2} + \frac{3}{4} = \frac{2\sqrt{3}+3}{4}.$$

$$y - \frac{2\sqrt{3}+3}{4} = -1\left(x - \frac{2+\sqrt{3}}{4}\right)$$

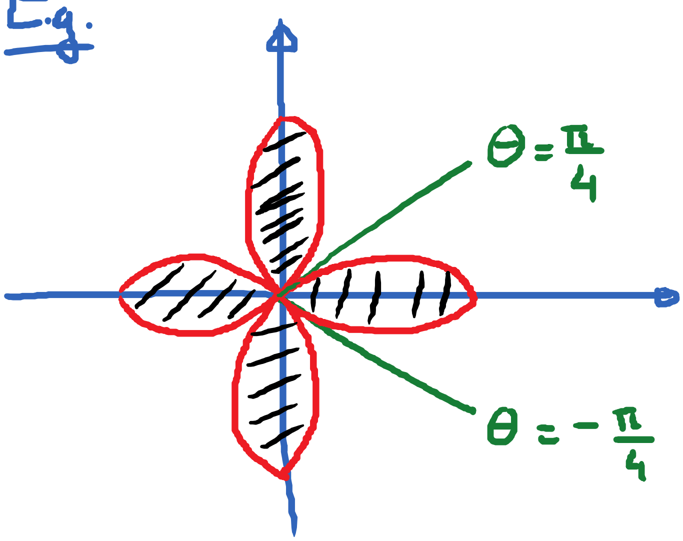
Areas with polar curves:



$R = f(\theta)$; $a \leq \theta \leq b$

$$\text{Shaded Area} = \int_a^b \frac{1}{2}R^2\,d\theta = \frac{1}{2}\int_a^b [f(\theta)]^2\,d\theta.$$

E.g.



$R = \cos(2\theta)$; $0 \leq \theta \leq 2\pi$

Area of 1 petal

$$= \frac{1}{2}\int_{-\pi/4}^{\pi/4} [\cos(2\theta)]^2\,d\theta.$$

$$\frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2(2\theta)\,d\theta \overset{?}{=} \frac{1}{2}\int_{-\pi/4}^{\pi/4} \frac{1+\cos(4\theta)}{2}\,d\theta$$

power reduction formula

$$= \frac{1}{4}\int_{-\pi/4}^{\pi/4} \left[1+\cos(4\theta)\right] d\theta$$

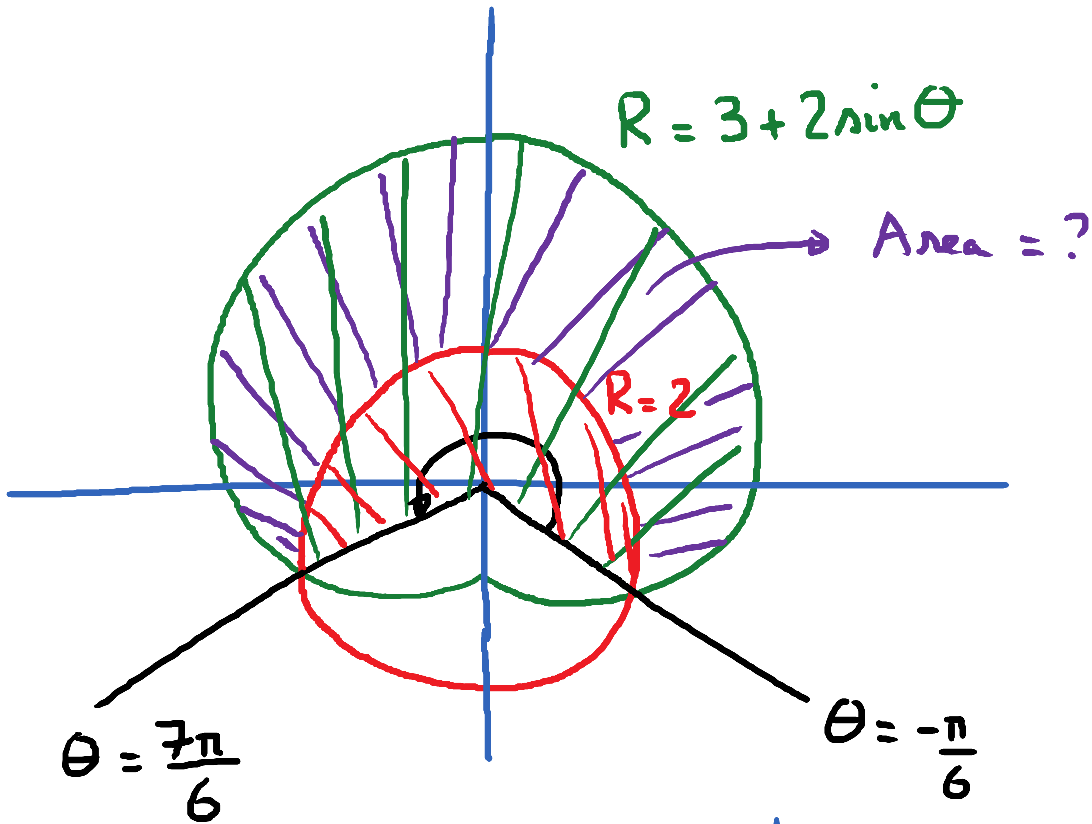
$$= \frac{1}{4}\cdot\left(\theta + \frac{\sin(4\theta)}{4}\right)\Bigg|_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{4}\left[\frac{\pi}{4} + \frac{\sin(\pi)}{4} - \left(-\frac{\pi}{4} + \frac{\sin(-\pi)}{4}\right)\right]$$

$$= \frac{1}{4}\cdot\left(\frac{\pi}{2}\right) = \boxed{\frac{\pi}{8}}$$

Area of whole thing $= 4\cdot\frac{\pi}{8} = \boxed{\frac{\pi}{2}}$.

Ex.



$$\frac{1}{2}\int_{-\pi/6}^{7\pi/6}(3+2\sin\theta)^2\,d\theta - \frac{1}{2}\int_{-\pi/6}^{7\pi/6}(2)^2\,d\theta$$

$$= \frac{1}{2}\cdot\int_{-\pi/6}^{7\pi/6}\left[(3+2\sin\theta)^2 - 4\right]d\theta$$

$$= \frac{1}{2}\int_{-\pi/6}^{7\pi/6}\left(9 + 12\sin\theta + 4\sin^2\theta - 4\right)d\theta$$

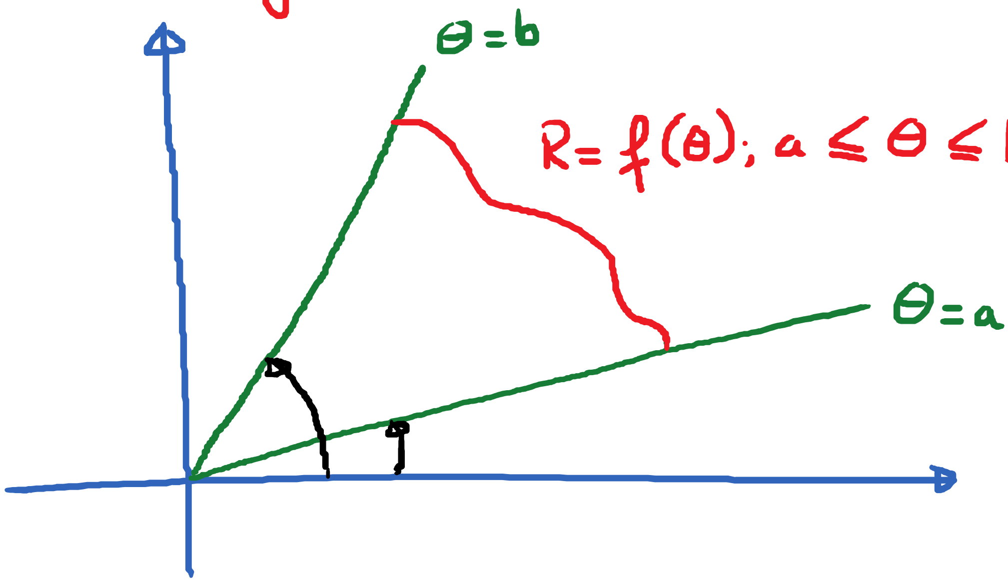
$$= \frac{1}{2}\int_{-\pi/6}^{7\pi/6} \left(5 + 12\sin\theta + 4\cdot\frac{1-\cos(2\theta)}{2} \right) d\theta$$

$$= \frac{1}{2}\int_{-\pi/6}^{7\pi/6} \left(7 + 12\sin\theta - 2\cos(2\theta) \right) d\theta$$

$$= \frac{1}{2}\cdot\left(7\theta - 12\cos\theta - \sin(2\theta) \right)\Big|_{-\pi/6}^{7\pi/6}$$

$$= \frac{11\sqrt{3}}{2} + \frac{14\pi}{3}.$$

Arc lengths of Polar Curves.



$R = f(\theta);\ a \le \theta \le b.$ Length of curve = ?

$$\boxed{L = \int_a^b \sqrt{R^2 + \left(\frac{dR}{d\theta}\right)^2}\, d\theta}$$

$$L = \int_a^b \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$$

E.g. Compute the length of the curve $R = \cos\theta$; $0 \le \theta \le \pi/2$.

$$L = \int_0^{\pi/2} \sqrt{\cos^2\theta + (-\sin\theta)^2}\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{\cos^2\theta + \sin^2\theta}\, d\theta = \int_0^{\pi/2} d\theta = \theta \Big|_0^{\pi/2} = \boxed{\pi/2}.$$

E.g. Find the length of the curve $R = 1 + \sin\theta$.

$0 \le \theta \le \frac{\pi}{2}$.

$$L = \int_0^{\pi/2} \sqrt{(1+\sin\theta)^2 + (\cos\theta)^2}\, d\theta$$

$$L = \int_0^{\pi/2} \sqrt{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta}\, d\theta$$

$$L = \int_0^{\pi/2} \sqrt{2 + 2\sin\theta}\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{2} \cdot \sqrt{1 + \sin\theta}\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{2} \cdot \frac{\sqrt{1+\sin\theta}}{1} \cdot \frac{\sqrt{1-\sin\theta}}{\sqrt{1-\sin\theta}} \, d\theta$$

$$= \sqrt{2} \int_0^{\pi/2} \frac{\sqrt{(1+\sin\theta)(1-\sin\theta)}}{\sqrt{1-\sin\theta}} \, d\theta$$

$$= \sqrt{2} \int_0^{\pi/2} \frac{\sqrt{1-\sin^2\theta}}{\sqrt{1-\sin\theta}} \, d\theta$$

$$= -\sqrt{2} \int_0^{\pi/2} \frac{\overbrace{(-\cos\theta \, d\theta)}^{du}}{\sqrt{\underbrace{1-\sin\theta}_{u}}}$$

Let $u = 1 - \sin\theta$

$du = -\cos\theta \, d\theta$

$$= -\sqrt{2} \int_1^0 \frac{du}{\sqrt{u}}$$

$\theta = 0 \longrightarrow u = 1 - \sin(0)$
$= 1$

$\theta = \frac{\pi}{2} \rightarrow u = 1 - \sin\left(\frac{\pi}{2}\right)$
$= 0$

$$= \sqrt{2} \int_0^1 \frac{du}{\sqrt{u}}.$$

$$= \sqrt{2}\int_0^1 u^{-1/2}\,du = \sqrt{2}\cdot \frac{u^{1/2}}{1/2}\Bigg|_0^1 = 2\sqrt{2}\,u^{1/2}\Big|_0^1$$

$$= \boxed{2\sqrt{2}}\,.$$

$$R = e^{4\theta}\ ; \quad 0 \leq \theta \leq \pi$$

$$L = \int_0^{\pi} \sqrt{\left(e^{4\theta}\right)^2 + 16e^{8\theta}}\ d\theta$$

$$= \int_0^{\pi} \sqrt{e^{8\theta} + 16e^{8\theta}}\ d\theta$$

$$= \int_0^{\pi} \sqrt{17e^{8\theta}}\ d\theta = \sqrt{17}\cdot\int_0^{\pi} e^{4\theta}\,d\theta$$

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$$= \sqrt{17} \cdot \left(\frac{e^{4\theta}}{4} \right) \Bigg|_0^{\pi} = \sqrt{17} \left(\frac{e^{4\pi}}{4} - \frac{1}{4} \right)$$

$$= \frac{\sqrt{17}}{4} \left(e^{4\pi} - 1 \right)$$

$$R = 6\sqrt{1 + \cos(2\theta)} \ ; \ 0 \le \theta \le \frac{\pi}{4}$$

$$L = \int_0^{\pi/4} \sqrt{36\left(1 + \cos(2\theta)\right) + \frac{36 \sin^2(2\theta)}{1 + \cos(2\theta)}}$$

$$\left[6\left(1 + \cos(2\theta)\right)^{1/2} \right]' = 6 \cdot \frac{1}{2} \cdot \left(1 + \cos(2\theta)\right)^{-1/2} \cdot \left(-2\sin(2\theta)\right)$$

$$= -6\left(1 + \cos(2\theta)\right)^{-1/2} \cdot \sin(2\theta)$$

$$= \frac{-6 \cdot \sin(2\theta)}{\sqrt{1 + \cos(2\theta)}}$$