#1

$$\int \cos(9x)\cos(4x)\,dx$$

$$= \int \frac{1}{2}\left[\cos(5x) + \cos(13x)\right] dx$$

$$= \frac{1}{2}\int\left(\cos(5x) + \cos(13x)\right) dx$$

$$= \frac{1}{2}\cdot\left[\frac{\sin(5x)}{5} + \frac{\sin(13x)}{13}\right] + C$$

#2

$$\int \frac{x^3}{\sqrt{x^2+4}}\,dx$$

Let $x = 2\tan\theta$; $dx = 2\sec^2\theta\,d\theta$

$$\int \frac{8\tan^3\theta}{\sqrt{4\tan^2\theta+4}}\cdot 2\sec^2\theta\,d\theta$$

$$= \int \frac{8\tan^3\theta}{\sqrt{4(\tan^2\theta+1)}}\cdot 2\sec^2\theta\,d\theta$$

$$= \int \frac{8\tan^3\theta}{\cancel{2}\cancel{\sec\theta}}\cdot \cancel{2}\sec^{\cancel{2}}\theta\,d\theta$$

$$= 8\int \tan^3\theta\cdot\sec\theta\,d\theta$$

$$= 8\int \tan^2\theta\cdot\boxed{\tan\theta\sec\theta\,d\theta}$$

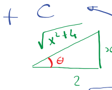
$$= 8\cdot\int(\sec^2\theta - 1)\cdot\underbrace{\tan\theta\sec\theta\,d\theta}_{du}$$

$u = \sec\theta \longrightarrow du = \sec\theta\tan\theta\,d\theta$

$$= 8\cdot\int(u^2-1)\,du = 8\cdot\frac{u^3}{3} - 8u + C$$

$$= 8\cdot\frac{\sec^3\theta}{3} - 8\sec\theta + C$$

$x = 2\tan\theta \longrightarrow \tan\theta = \frac{x}{2}$



$\sec\theta = \frac{\sqrt{x^2+4}}{2}$

$$\frac{8}{3}\left(\frac{\sqrt{x^2+4}}{2}\right)^3 - 8\cdot\frac{\sqrt{x^2+4}}{2} + C$$

$$= \frac{(x^2+4)^{3/2}}{3} - 4\sqrt{x^2+4} + C.$$

#3

$$\int \frac{\overbrace{e^t\,dt}^{du}}{\underbrace{e^{2t}}_{u^2} - 7\underbrace{e^t}_{u} + 6}$$

Let $u = e^t$. $du = e^t\,dt$

$$\int \frac{du}{u^2 - 7u + 6} = \int \frac{du}{(u-1)(u-6)}$$

Partial Fractions Decomposition:

$$\frac{1}{(u-1)(u-6)} = \frac{A}{u-1} + \frac{B}{u-6}$$

$$1 = A(u-6) + B(u-1)$$

Put $u = 6$: $1 = 5B \longrightarrow B = 1/5$

Put $u = 1$: $1 = -5A \longrightarrow A = -1/5$

$$\int \frac{du}{(u-1)(u-6)} = -\frac{1}{5}\int\frac{du}{u-1} + \frac{1}{5}\int\frac{du}{u-6}$$

$$= -\frac{1}{5}\ln|u-1| + \frac{1}{5}\ln|u-6|$$

$$-\frac{1}{5}\ln|e^t-1| + \frac{1}{5}\ln|e^t-6|$$

#5

$$\int_0^\infty \frac{4(1+\tan^{-1}x)}{1+x^2}\,dx = \lim_{t\to\infty} 4\int_0^t \frac{\overbrace{1+\tan^{-1}x}^{u}}{\underbrace{1+x^2}_{du}}\,dx$$

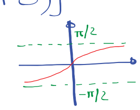
Let $u = 1 + \tan^{-1}x$. $du = \frac{1}{1+x^2}\,dx$

$$\int u\,du = \frac{u^2}{2} = \frac{(1+\tan^{-1}x)^2}{2}.$$

$$\lim_{t\to\infty} 4\cdot\left.\frac{(1+\tan^{-1}x)^2}{2}\right|_0^t$$

$$\lim_{t\to\infty}\left[2(1+\tan^{-1}t)^2 - 2(1+0)\right]$$

$$\lim_{t\to\infty}\left[2(1+\tan^{-1}t)^2 - 2\right]$$



$$2\left(1+\frac{\pi}{2}\right)^2 - 2 = 2\left[1 + 2\cdot1\cdot\frac{\pi}{2} + \frac{\pi^2}{4}\right] - 2$$

$$= 2\left[1 + \pi + \frac{\pi^2}{4}\right] - 2$$

$$= 2\pi + \frac{\pi^2}{2} = 2\pi\left(1 + \frac{\pi}{4}\right)$$

(6) $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots$

$a_n = \frac{(-1)^{n+1}}{n^2}$; $n=1 \to 1$

$n=2 \to \frac{-1}{4}$

$n=3 \to \frac{1}{9}$ ....

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(7) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2 - 1n + 3n^4}{5n^4 - 2n^3 + 2}$

Degree of top $= 4 =$ Degree of bottom.

So, $\lim_{n\to\infty} a_n = \frac{\text{leading coeff. top}}{\text{leading coeff. bottom}} = \frac{3}{5}$.

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(8) $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{9}{7^n} = \sum_{n=1}^{\infty} (-1)\cdot(-1)^n \cdot \frac{9}{7^n}$

$= \sum_{n=1}^{\infty} (-9)\cdot\left(-\frac{1}{7}\right)^n = (-9)\cdot \underbrace{\sum_{n=1}^{\infty}\left(-\frac{1}{7}\right)^n}_{\text{geometric series}}$

common ratio $= -\frac{1}{7}$

Sum $= (-9)\cdot \frac{-\frac{1}{7}}{1-\left(-\frac{1}{7}\right)} = \boxed{\frac{9}{8}}$

(9) $\sum_{n=1}^{\infty} \underbrace{\cos\left(\frac{6}{n}\right)}_{a_n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos\left(\frac{6}{n}\right) = \cos(0) = 1 \neq 0$

The series diverges by the divergence test.

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(10) I. $a_n = n(\sin n + 1)$

Cannot use integral test for $\sum a_n$ b/c a_n is not always increasing (due to the sine function)

II. $a_n = \frac{1}{n^p + p}$. a_n is not always increasing. It depends on p.

III. $a_n = \frac{1}{n\sqrt{n}}$. ~~Wh~~ We can apply the integral test for $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

(11) We can use the trig sub:

$t = \sqrt{18}\sin\theta$ or it can be rewritten as:

$t = 2\sqrt{2}\sin\theta$.

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(12) $\int \frac{8x^2 + x + 63}{x^3 + 9x}\,dx$

$= \int \frac{8x^2 + x + 63}{x(x^2+9)}$.

The form for the partial fractions decomposition is:

$\frac{8x^2+x+63}{x(x^2+9)} = \frac{A}{x} + \frac{Bx+C}{x^2+9}$.

(13) Error estimate for Simpson's rule:

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$

K: upper bound for $|f^4(x)|$ on $[a,b]$

$f(x) = \frac{1}{x-1}$ here. $f'(x) = -\frac{1}{(x-1)^2}$

$f''(x) = \frac{2}{(x-1)^3}$; $f^{(3)}(x) = \frac{-6}{(x-1)^4}$

$f^{(4)}(x) = \frac{24}{(x-1)^5}$.

On $[2,4]$, the maximum value of $|f^{(4)}(x)|$ is

$\frac{24}{(2-1)^5} = 24$.

We want: $\frac{24\cdot(4-2)^5}{180n^4} \leq 10^{-4}$

$\longleftrightarrow \frac{64}{15n^4} \leq 10^{-4} = \frac{1}{10^4}$

$$\longleftrightarrow \frac{15n^4}{64} \geqslant 10^4$$

(take the reciprocal of both sides & switch direction of inequality)

$$\longleftrightarrow n^4 \geqslant 10^4 \cdot \frac{64}{15}$$

$$\longleftrightarrow n \geqslant \sqrt[4]{10^4 \cdot \frac{64}{15}} \approx 14.4.$$

So $n = 15$ should be sufficient.

(14) $\int_{-1}^{1} \frac{1}{x\ln|x|}\,dx$.

This integral is improper because the function is discontinuous at $x = 0$

$$\int_{-1}^{1} \frac{1}{x\ln|x|}\,dx = \int_{-1}^{0} \frac{dx}{x\ln|x|} + \int_{0}^{1} \frac{dx}{x\ln|x|}$$

$$\int_{-1}^{0} \frac{dx}{x\ln|x|} = \lim_{t\to 0^-} \int_{-1}^{t} \frac{dx}{x\ln|x|} \; ; \; \begin{cases} u = \ln|x| \\ du = \frac{1}{x}\,dx \end{cases} \quad \int \frac{du}{u} = \ln|u|$$

$$= \lim_{t\to 0^-} \ln|\ln|x|| \Big|_{-1}^{t}$$

$$= \lim_{t\to 0^-} \ln|\ln|t|| - \underbrace{\ln|\ln|-1||}$$

$$= \infty$$

Since one of the integrals diverges, the original integral diverges.

(16) $\int 7\sec^4 x\,dx$

$$= 7\int \sec^2 x \, \sec^2 x\,dx$$

u^2

du

$$= 7\int (1 + \tan^2 x)\sec^2 x\,dx$$

$u = \tan x \longrightarrow du = \sec^2 x\,dx$

$$7\int (1 + u^2)\,du$$

$$= 7\cdot\left(u + \frac{u^3}{3}\right) + C$$

$$= 7\cdot\left(\tan x + \frac{\tan^3 x}{3}\right) + C.$$

(17) $\sum_{n=0}^{\infty} (-1)^n \cdot \left(\frac{x-3}{7}\right)^n$

$$= \sum_{n=0}^{\infty} \left(-\frac{x-3}{7}\right)^n$$

This is a geometric series with common ratio equal to $-\frac{x-3}{7}$.

For this series to converge, we must have $\left|-\frac{x-3}{7}\right| < 1$. So, $\left|\frac{x-3}{7}\right| < 1$

So $|x-3| < 7$. So, $-7 < x-3 < 7$

So, $-4 < x < 10$.

Values of x for which series converges: any value in $(-4, 10)$

(18) $\sum_{n=1}^{\infty} \frac{\cos(1/n)}{n^2}$

$f(n) = \frac{\cos(1/n)}{n^2}$

→ positive ($\frac{1}{n}$ is in $[0, \frac{\pi}{2}]$ for all $n \geq 1$)

→ decreasing:

($f'(n) = \frac{\sin(\frac{1}{n}) - 2n \cdot \cos(\frac{1}{n})}{n^4}$

eventually $f'(n) < 0$)

→ continuous on $[1, \infty)$

So, f is positive, decreasing (eventually) and continuous. We can apply the Integral Test here.

$\int_1^{\infty} \frac{\cos(\frac{1}{x})}{x^2} dx$ let $u = \frac{1}{n}$. $du = -\frac{1}{n^2} dx$

$-\int \cos(u)\, du = -\sin(u)$

$\longrightarrow -\sin\left(\frac{1}{x}\right)\Big|_1^{\infty} = \sin(1)$.

So integral converges. So, series converges.

(19) $\sum_{n=1}^{\infty} \frac{4\sqrt{n}}{10n^{3/2} - 7n + 2} \longrightarrow a_n$

The terms behave (when n is large) like:

$\frac{4\sqrt{n}}{10n^{3/2}} = \frac{2n^{1/2}}{5n^{3/2}} = \frac{2}{5n}$.

So, we can apply the limit comparison test with $\sum_{n=1}^{\infty} \frac{2}{5n}$ b_n

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{4\sqrt{n}}{10n^{3/2} - 7n + 2}}{\frac{2}{5n}}$.

$= \lim_{n\to\infty} \frac{20n\sqrt{n}}{20n^{3/2} - 14n + 4} = 1$

(degree top = degree bottom $= \frac{3}{2}$)

Since limit $= 1$ (finite, $< \infty$). $\sum a_n$ behaves like $\sum b_n$. $\sum b_n = \sum \frac{2}{5n}$ diverges (p-series $p = 1$). So, original series diverges.

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(20) $\sum_{n=1}^{\infty} \left(\frac{n}{9n+8}\right)^n$.

We have: $\frac{n}{9n+8} < \frac{n}{9n} = \frac{1}{9}$.

So, $\left(\frac{n}{9n+8}\right)^n < \left(\frac{1}{9}\right)^n$

So, $\sum_{n=1}^{\infty} \left(\frac{n}{9n+8}\right)^n < \sum_{n=1}^{\infty} \left(\frac{1}{9}\right)^n$

The series $\sum_{n=1}^{\infty} \left(\frac{1}{9}\right)^n$ converges because the ratio is $\frac{1}{9} < 1$.

So, the original series converges.

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