HONORS CALCULUS I - PROJECT III - SOMEWHERE, WITHIN THE RAINBOW, ... IS SOME CALCULUS

ABSTRACT. The rainbow is one of the most magnificent natural phenomenon. Rainbows result from reflection, refraction and dispersion of light in falling water droplets. In this project, we will explore some of the mathematics behind this splendid phenomenon.

1. Fermat, Snell, Reflection and Refraction

When light from the sun encounters a raindrop, part of it is reflected. The rest of the light enters the raindrop and is refracted at the surface of the raindrop. It then hits the back of the raindrop and some of it is reflected off the back. When the latter part hits the surface again, some is internally reflected and some is refracted as it exits the raindrop (See Figure 1). This process is fundamental in the formation of the rainbow. Hence, our first step is to understand the law of reflection and refraction of light.

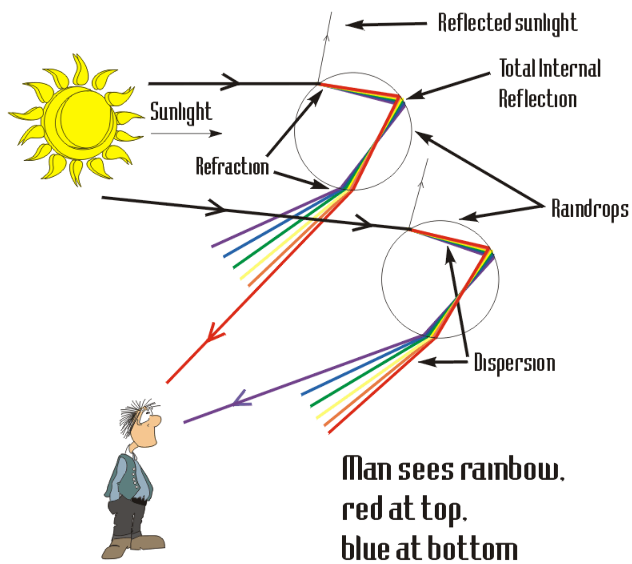

FIGURE 1. Rainbow Formation - from http://www.rebeccapaton.net/

Law of Reflection. A ray of light approaches and reflects off a flat mirror (or surface), then the angle of incidence equals the angle of reflection.

The Law of Reflection can be deduced from Fermat's principle, which says that light is very smart, it will always follow a path that minimizes the total travel time. In what follows, we shall go through this deduction of the Law of Reflection from Fermat's principle.

Consider a light ray starting at a point P, approaching a point R on a flat mirrow and passing through Q after reflecting off the mirror (See figure 2). Our goal is to show that the angle of incidence α is equal to the angle of of reflection β.

Since the speed of light is constant, the total travel time is minimized when R is positioned so that the path PRQ has minimum length.

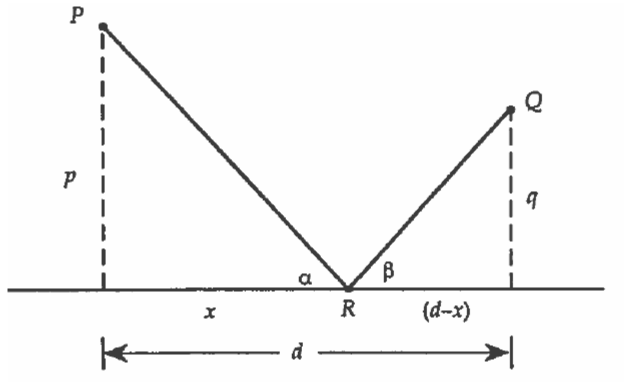

FIGURE 2. Reflection

Question 1. From figure 2, write down the formula for the function $L(x)$ which gives the length of the path PRQ as a function of x, the position of R. (Hint: $L(x) = PR + RQ$, find PR and RQ in terms of x. We consider d, p, q as constants here.)

Question 2. The length function $L(x)$ is minimized at a point x where $L'(x) = 0$. Find $L'(x)$ and set it equal to zero to find x in terms of the constants p, d and q. Show that when $L'(x) = 0$, we have $\cos(\alpha) = \cos(\beta)$.

Question 3. We should really check that the value of x in Question 2 actually gives a minimum of $L(x)$ by taking the second derivative. So, find the second derivative of $L(x)$ and explain why the x in Question 2 does give a minimum of $L(x)$.

Answering Question 1, 2 and 3 establishes that the angle of incidence α is equal to the angle of reflection β, hence, we have deduced the Law of Reflection (because $\cos\alpha = \cos\beta$ and $0 < \alpha, \beta < 90°$).

When light travels from air to water, it slows down because water is more dense than air is. When light slows down, its path bends toward the normal line and so the angle of incidence is greater than the angle of refraction. (See figure 3).

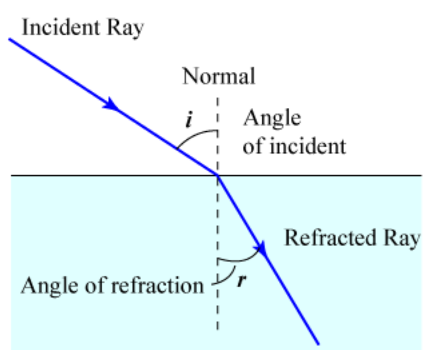

FIGURE 3. Refraction - from http://www.one-school.net

The scientist Willebrord Snellius did experiments with this phenomenon and came up with:

Law of Refraction. The ratio of the sine of the angle of incidence to the sine of the angle of refraction is a constant.

In what follows, we will prove this law which has come to be known as **Snell's Law**.

Consider a light ray starting a point P in air, crossing the boundary between air and water at a point R and passing through the point Q in water (See figure 4). The angle of incidence and angle of refraction in figure 4 are α and β, respectively. Let c_a and c_w denote the speed of light in air and in water, respectively. We shall prove that

$$\frac{\sin\alpha}{\sin\beta} = \frac{c_a}{c_w} = \text{ constant } k.$$

Here $k = \frac{c_a}{c_w}$ is a constant which is called the *index of refraction for water*. The constant k is approximately $\frac{4}{3}$.

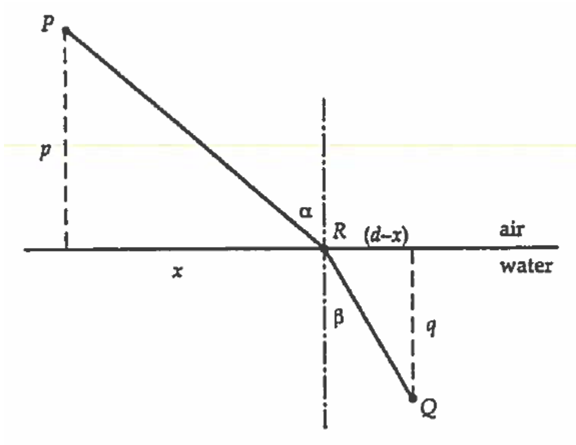

FIGURE 4. Refraction

Question 4. From figure 4, write down the formula for the function $T(x)$ which gives the time it takes the light ray to travel from P to Q as a function of x, the position of R. (Hint: $T(x)$ equals to the sum of the time it takes the ray to travel from P to R and from R to Q. From P to R the light ray travels in air with speed c_a and from R to Q it travels in water with speed c_w. We consider c_a, c_w, p, q and d as constants here.)

Question 5. By Fermat's principle, the light ray follows the path that minimizes the travel time. Hence, R is positioned at x such that $T(x)$ is minimized. Show that when $T(x)$ is minimized, we have

$$\frac{\sin\alpha}{\sin\beta} = \frac{c_a}{c_w}.$$

(Hint: $T(x)$ is minimized at x with $T'(x) = 0$. Hence, find the first derivative of $T(x)$, set it equal to zero and use that equality to establish the equality you are trying to prove.)

2. The Rainbow Angle and Colors

Now we understand the law of reflection and refraction, to understand the formation of the rainbow, we need to keep track of the reflections and refractions of a ray of sunlight entering a single raindrop.

Consider figure 5. A ray of sunlight is entering a spherical raindrop at the point A. It is refracted, continues through the drop and strikes the other side at the point B. At A, α is the angle of incidence and β is the angle of refraction and the ray has been deflected by an amount $\alpha - \beta$. At B, the ray is reflected and by the Law of Reflection, the angle of incidence is equal to the angle of reflection, both of which is equal to β. Hence, the ray is deflected by an additional amount $180° - 2\beta$. Then it continues and hits the surface of the drop at C. Part of it will be reflected but we are interested in the part of it that is refracted and exits the drop. The reason is because every time the ray hits the surface, its intensity decreases. Thus, rays that strikes the drop's surface only a few times will be the brightest ones. Now at C, since the angle of incidence is α, the angle of refraction must be β and the ray once again is deflected, in an amount $\alpha - \beta$. In overall, if we denote the angle of deflection by $D(\alpha)$, then

$$D(\alpha) = (\alpha - \beta) + (180° - 2\beta) + \alpha - \beta = 180° + 2\alpha - 4\beta.$$

Now D is a function of both α and β. However, we can consider β as a function of α thanks to the Law of Refraction. Hence, we can then consider $D(\alpha)$ as a function of α. We are interested in find the incidence angle α that gives the minimum deflection $D(\alpha)$. The reason is because rays coming from the direction of minimum deflection will appear the brightest since they are spread out the least.

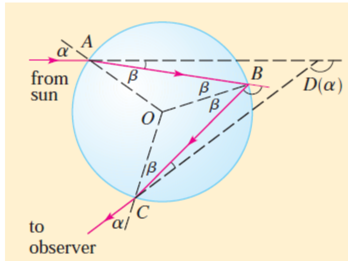

FIGURE 5. Rainbow formation

Question 6. Consider β as a function of α and apply implicit differentiation to the relation $\dfrac{\sin\alpha}{\sin\beta} = k$ to find $\dfrac{d\beta}{d\alpha}$. (Note that $k = \frac{c_a}{c_w} \approx \frac{4}{3}$ is a constant, the index of refraction).

Question 7. Find $D'(\alpha)$, the first derivative of $D(\alpha)$ with respect to α, and substitute the expression for $\dfrac{d\beta}{d\alpha}$ in Question 6 into the expression for $D'(\alpha)$. Set $D'(\alpha)$ equal to zero and shows that $D'(\alpha) = 0$ implies that

$$\frac{\cos\alpha}{\cos\beta} = \frac{k}{2}. \tag{1}$$

Question 8. Square both sides of Equation 1. Use the identity $\cos^2\beta = 1 - \sin^2\beta$ and the relation between $\sin\alpha$ and $\sin\beta$ to show that Equation 1 implies that

$$\cos\alpha = \sqrt{\frac{k^2-1}{3}}. \tag{2}$$

Equation 2 gives us an expression for the cosine of the incidence angle with minimum deflection. In particular, for $k = \frac{4}{3}$, $\cos\alpha \approx 0.5063$ and so $\alpha \approx 59.6°$. At this incidence angle, the deflection angle is $D(59.6) \approx 138°$. This means that many rays with incidence angle $\alpha \approx 59.6°$ become deflected by the same amount. These rays will appear the brightest and this is where the rainbow appears. A ray whose incidence angle is $\approx 59.6°$ is called a *rainbow ray*. The angle of elevation from the observer up to the highest point on the rainbow is $42° = (180° - 138°)$ and this is called the *rainbow angle*. See figure 6.

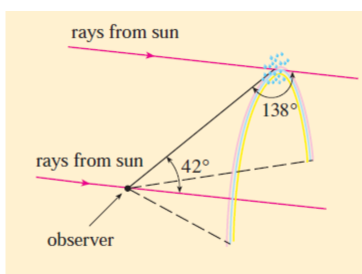

FIGURE 6. Rainbow Angle

The analysis from Question 6 through Question 8 explains the formation of the rainbow. But where do the different colors come from?

Light exhibits properties of both waves and particles. Only light with certain wavelengths are visible to the human eye. Visible light usually has wavelengths in the range of 400-700 nanometers (nm). Light with a wavelength of about 650 nm is perceived as red, 400 nm is perceived as blue, 445 nm is perceived as indigo, etc. by the human eye. The index of refraction is actually different for each color and this effect is called *dispersion*. The refractive index for red light traveling from air to water is $k \approx 1.3318$, for violet light is $k \approx 1.3435$.

Question 9. Use the refractive index for red light and violet light to show that the rainbow angle for the red bow is about 42.3° and for the violet bow is about 40.6°.

3. The Missing Rainbow

Perhaps you have seen a double rainbow. The secondary bow is above the primary one and appears fainter, and the colors in the secondary bow appear in reverse order from those in the primary bow (See figure 7).

Figure 7. Double Rainbow

In the previous section, we traced a ray of light that was reflected once after it entered the raindrop. The secondary bow results from rays that have two internal reflections in the raindrop. More reflections will reduce the intensity of the rays. That is why the secondary bow appears fainter.

Consider figure 8. Here we trace a ray that results in a secondary bow. Note that the ray is reflected twice in the raindrop at B and at C.

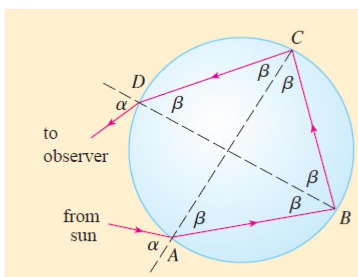

Figure 8. A ray with two internal reflections

Question 10. Use figure 8 to explain why the total deflection of the ray is given by the formula

$$D_2(\alpha) = 360^\circ + 2\alpha - 6\beta.$$

Note that a 360° deflection means that the ray continues in the same direction in which it started. Hence, we can disregard the 360° in the formula for $D_2(\alpha)$. Moreover, when we traced the ray in the previous section, we considered a clockwise deflection (See figure 5). Now, the deflection in figure 8 is counterclockwise and to keep it consistent, we will make it clockwise by switching the signs and consider a new deflection function, which, by abuse of notation, we still call $D_2(\alpha)$ and so

$$D_2(\alpha) = 6\beta - 2\alpha.$$

In Question 6 through 8 in the previous section, we found $\dfrac{d\beta}{d\alpha}$ and use it to find $D'(\alpha)$. Then we set $D'(\alpha)$ equal to zero to show that at the critical point, the incidence angle α satisfies $\cos\alpha = \sqrt{\dfrac{k^2-1}{3}}$. This has allowed us to find the rainbow angle for the primary bow.

Question 11. Follow a similar strategy to find $D_2'(\alpha)$, set it equal to zero and show that the critical point of $D_2(\alpha)$ satisfies

$$\cos\alpha = \sqrt{\frac{k^2-1}{8}}.$$

Note that if you find $D''(\alpha)$ and $D_2''(\alpha)$, you will see that $D''(\alpha) > 0$ and $D_2''(\alpha) < 0$. Hence, the critical point in the previous section is the minimum deflection angle whereas the one here is the maximum deflection angle.

Question 12. Use the refractive index $k = \frac{4}{3}$ to show that the maximum deflection angle for the secondary bow is $D_2(\alpha) \approx 129°$.

As a result, the rainbow angle for the secondary bow is approximately 51°. See figure 9 which shows the rainbow angle for both the primary bow (calculated in the previous section) and the secondary bow.

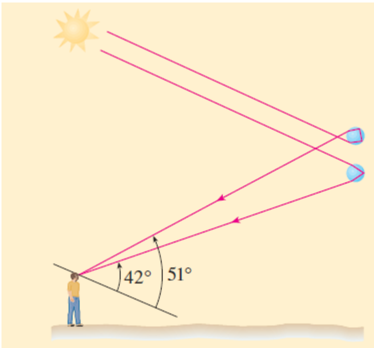

FIGURE 9. Double rainbow

Question 13. Use the refractive index $k \approx 1.3318$ for red light and $k \approx 1.3435$ for violet light to show that the maximum deflection angle for red light is $D_2(\alpha) \approx 129.424°$ and for violet light is $D_2(\alpha) \approx 126.395°$.

As a result, the rainbow angle for red light is smaller than that of violet light. This shows that the colors in the secondary bow appear in reverse order from those in the primary bow, as you can see in figure 7.

Question 14 - Further Research. This part is optional but you can earn up to 5 extra credit points if you write a 200-word-essay to fully address all of the following questions:

- In optics, what is dispersion?
- Describe Newton's experiment when he discovered the phenomenon of dispersion.
- What causes dispersion?
- Describe Cauchy's equation on the relationship between refractive indices and wavelengths. In particular, explain what each variable in the equation means.
- Use Cauchy's equation to explain why the refractive indices decrease with increasing wavelengths for visible light.