HONORS CALCULUS I - PROJECT II - TANGENTS. TANGENTS EVERYWHERE

1. Bézier Curves

Bézier curves are fundamental in many computer graphics programs such as Adobe Illustrator, Corel Draw and Inkscape. The mathematics behind the beautifully designed Chords bridge in Jerusalem (see Figure 1) also involves Bézier curve.

Figure 1. Chords Bridge

Figure 2 below shows three Bézier curves. A quadratic curve and two cubic curves. A quadratic Bézier curve is determined by three control points P_0, P_1 and P_2. The curve starts at P_0, ends at P_2 and its tangent lines at P_0 and P_2 intersect at the point P_1.

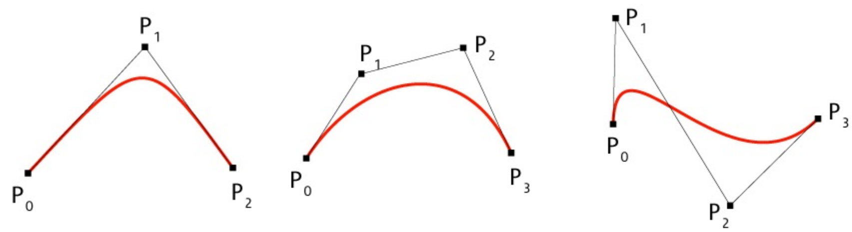

Figure 2. Quadratic and Cubic Bézier Curves

A cubic Bézier curve is determined by four control points P_0, P_1, P_2 and P_3. The cubic curve starts at P_0, ends at P_3. When it leaves P_0, it is heading towards P_1 and when it arrives at P_3, it is coming from the direction of P_2. In this project, we will study some basic Bézier curves and their properties.

First we recall some material from Precalculus.

Suppose that t is a variable taking values from an interval I. Let x and y be both given as functions of t (called a *parameter*) by the equations $x = x(t)$, $y = y(t)$ (called *parametric equations*). Each value of t determine a point $(x(t), y(t))$ in the coordinate plane. As t varies, the point $(x(t), y(t))$ varies and traces out a plane curve, called a *parametric curve*. For example,

Question 1. What curve is represented by the following parametric equations? Explain your reasoning.

$$x = \cos(t), y = \sin(t), 0 \le t \le 2\pi.$$

Question 2. Let $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$ be two points in the plane such that the line passing through P_0 and P_1 is not vertical. Show that the parametric equations for the (directed) line segment from P_0 to P_1 are

$$x = x_0 + t(x_1 - x_0) \text{ and } y = y_0 + t(y_1 - y_0) \text{ where } 0 \le t \le 1.$$

(Hint: to get an idea of how to prove this, think about a special example. Suppose that P_0 is the origin $(0,0)$ and $P_1 = (2,3)$. Then the above statement says that the parametric equations $x = 2t$, $y = 3t$, $0 \leq t \leq 1$ will parameterize the line segments from the origin to $(2,3)$. Is this true? If we choose an arbitrary number t in the interval $[0,1]$ and plug it into the equations for x and y, does that point (x,y) belong to the line segment? Conversely, if you pick some random point (x,y) on that line segment, can you find t such that $x = 2t$ and $y = 3t$. After you think through this special case, see if you can turn your idea into a proof of the general statement.)

Before we move on, let's agree on some conventions so that the notations in what follows will not get too cumbersome. From now on, if I have a point $P = (x,y)$ and a number t, the notation tP means the point whose coordinates is (tx, ty) and I shall write $tP = (tx, ty)$. If I have two points $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$, I shall use the notation $P_0 \pm P_1$ to denote the points with coordinates $(x_0 \pm x_1, y_0 \pm y_1)$ and I shall write $P_0 \pm P_1 = (x_0 \pm x_1, y_0 \pm y_1)$.

Using these conventions, the two equations in Question 2 above can be combined into a single equation

$$(x,y) = (x_0, y_0) + t(x_1 - x_0, y_1 - y_0).$$

Better still, the above equation can be written in a more compact form as

$$P = P_0 + tP_1$$

where P is a point with coordinates (x,y).

Now, we get to the fun part. We will derive the formula for the quadratic Bézier curve determined by three points P_0, P_1 and P_2 from scratch.

The idea is to generate points on the curve using a divide and conquer method. We start by choosing a number t, $0 \leq t \leq 1$ ($t = .25$ in figure 3). Let Q_0 be the point on the directed line segment P_0P_1 and Q_1 be the point on the directed line segment P_1P_2 defined by

$$Q_0 = P_0 + t(P_1 - P_0) \text{ and } Q_1 = P_1 + t(P_2 - P_1) \tag{1}$$

Let B be the point on the directed line segment Q_0Q_1 defined by

$$B = Q_0 + t(Q_1 - Q_0) \tag{2}$$

See figure 3.

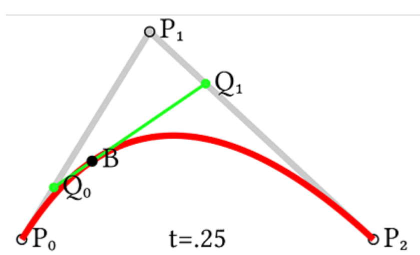

FIGURE 3. Points on Quadratic Bézier Curves

Then we let t vary, say $t = .26, .27, .28$, etc. When t varies, the points Q_0 and Q_1 will vary. As a result, B will vary and it will trace our a path which is the Bézier curve with the properties that we want.

Go to the link `https://i.stack.imgur.com/I6MjU.gif` to see an animation of the process.

We shall derive the set of parametric equations for the coordinates of the points B on the Bézier curve.

Question 3. Substitute the formula for Q_0 and Q_1 from equation 1 into the right hand side of equation 2 and simplify to show that

$$B = (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2. \tag{3}$$

Let's pause for a moment and try to see what equation 3 is saying. Equation 3 tells us how to generate the x and the y coordinates of any point B on the Bézier curve. In fact, suppose that $P_0 = (x_0, y_0)$,

$P_1 = (x_1, y_1)$, and $P_2 = (x_2, y_2)$, then equation 3 says that the x and the y coordinates of any point B on the Bézier curve are given by the pair of parametric equations

$$x = (1-t)^2 x_0 + 2t(1-t)x_1 + t^2 x_2 \text{ and } y = (1-t)^2 y_0 + 2t(1-t)y_1 + t^2 y_2 \text{ where } 0 \le t \le 1.$$

Of course, the computer programs do not use the parametric equations to generate the points, they use the divide and conquer method described above. But for the purpose of understanding the properties of Bézier curves, it is very useful to obtain these equations.

Question 4. Choose three specific points P_0, P_1 and P_2 that you like. Use equation 3 to find the parametric equations for the Bézier curve defined by your three points, that is, find explicitly the equation for the coordinate x and that for the coordinate y of an arbitrary point on a curve. Plot a couple of points on the curve by choosing several values of t in $[0, 1]$ and connect them to provide a rough sketch of your quadratic Bézier curve.

We'd better make sure that the formula for the curve in equation 3 actually defines a curve with the "nice" property described at the beginning of this section.

Question 5. Use the specific points you chose for Question 4 to prove the following about your Bèzier curve.

(1) When $t = 0$, the curve starts at P_0. When $t = 1$, the curve ends at P_2.
(2) Both of the tangent lines to the curve at P_0 and at P_2 pass through the point P_1.

Hint: we need to be careful with part (2). The slope of the tangent line to the curve, say at P_0, is $\frac{dy}{dx}$ at $x = x_0$. However, the issue here is that the curve is not described by a function $y = f(x)$. Both x and y are functions of t. It turns out that we have

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=0}.$$

So the slope at P_0 can be found by evaluating $\frac{y'(0)}{x'(0)}$. Similarly, you can find the slope at P_1.

A similar process can be used to obtain a formula for the cubic Bézier curve determined by four control points P_0, P_1, P_2 and P_3 (See figure 4 for a particular instance of the process and see the link `https://upload.wikimedia.org/wikipedia/commons/d/db/B%C3%A9zier_3_big.gif?1485969558963` for an animation of the whole process).

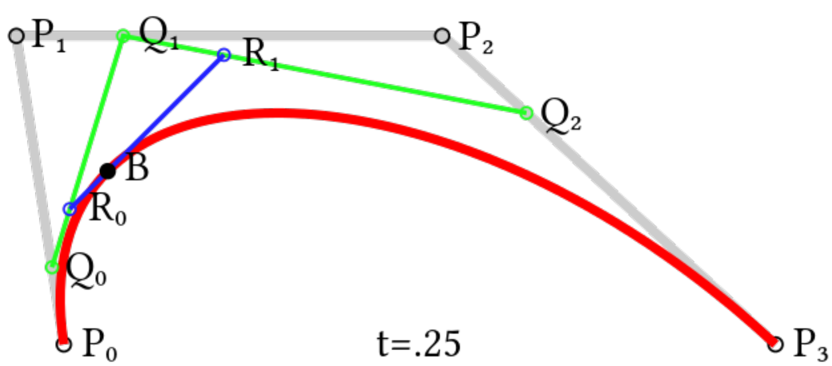

FIGURE 4. Cubic Bézier Curves

We will not go through this process here but the formula for the coordinates of the points B on the cubic curve is given by

$$B = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t)P_2 + t^3 P_3 \tag{4}$$

Question 6. Choose four specific points P_0, P_1, P_2 and P_3 that you like. Use equation 4 to find the parametric equations for the Bézier curve defined by your four points, that is, find explicitly the equation for the coordinate x and that for the coordinate y of an arbitrary point on a curve. Plot a couple of points on the curve by choosing several values of t in $[0, 1]$ and connect them to provide a rough sketch of your cubic Bézier curve.

Question 7. Use the specific points you chose for Question 6 to prove the following about your cubic Bèzier curve.

(1) When $t = 0$, the curve starts at P_0. When $t = 1$, the curve ends at P_3.
(2) The tangent line to the curve at P_0 passes through P_1 and the tangent line to the curve at P_3 passes through P_2.

What about a Bézier curve of degree 4, degree 5, etc.? What do the formulas for the coordinates points on those curves look like? See the link `https://upload.wikimedia.org/wikipedia/commons/0/0b/BezierCurve.gif?1485969726446` for the birth of a curve of fifth degree, a *quintic* Bézier curve.

We will make some useful observation here. Recall the binomial expansion for powers of $(a + b)$

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

Now, if you apply these expansions when $a = 1 - t$ and $b = t$, you will get

$$1 = [(1-t)+t]^2 = (1-t)^2 + 2t(1-t) + t^2$$
$$1 = [(1-t)+t]^3 = (1-t)^3 + 3t(1-t)^2 + 3t^2(1-t) + t^3$$

Notice the similarity between this pattern and the pattern of the formula for the points on the quadratic and cubic Bézier curves in equation 3 and equation 4.

Question 8. From the observation above, write down the formula for the coordinates of the points on the *quartic* Bézier curve determined by 5 control points $P_0, \ldots, P_4$. Write down the formula for the coordinates of the points on the *quintic* Bézier curve determined by 6 control points $P_0, \ldots, P_5$. In general, what is the formula for the n^{th}-degree Bézier curve determine by n control points $P_0, \ldots, P_n$?

You do not need to prove any of these formulas. However, it would be nice if you write a short paragraph describe what strategy you would use if you are to prove them.

2. Polynomial Approximation of Functions

Polynomial functions are among the simplest types of functions in mathematics. Yet, they can be used to approximate many complicated non-linear functions. In this section, we will study such an approximation. The polynomials that we will use to approximate various functions actually come from the formulas for the Bézier curves in the previous section.

First, note that the binomial expansion

$$1 = [t + (1-t)]^n = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k}$$

seems to play an important role in the formulas for Bézier curves. However, note that the polynomial on the right hand side is always 1 for every n, so we are going to twist it a little bit to make it more useful.

Let f be an arbitrary function which is defined and bounded on the interval $[0, 1]$. Let $n \geq 1$ be an arbitrary positive integer. We define $B_n(f)$ to be the following expression

$$B_n(f) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} f\left(\frac{k}{n}\right). \tag{5}$$

This looks like a complicated expression. Let's play with it a little bit.

Question 9. Suppose that f is the function $f(x) = x$. Then $f\left(\frac{k}{n}\right) = \frac{k}{n}$ and equation 5 becomes

$$B_n(f) = \sum_{k=0}^{n} \binom{n}{k} \frac{k}{n} t^k (1-t)^{n-k}.$$

Use the above formula to show that $B_1(f) = B_2(f) = B_3(f) = t$. Make a conjecture about the formula for $B_n(f)$. Can you prove your conjecture?

Question 10. Suppose that f is the function $f(x) = x^2$. Then $f\left(\frac{k}{n}\right) = \left(\frac{k}{n}\right)^2 = \frac{k^2}{n^2}$ and equation 5 becomes

$$B_n(f) = \sum_{k=0}^{n} \binom{n}{k} \frac{k^2}{n^2} t^k (1-t)^{n-k}.$$

Use the above formula to show that $B_1(f) = t$, $B_2(f) = \frac{t^2}{2} + \frac{t}{2}$, $B_3(f) = \frac{2t^2}{3} + \frac{t}{3}$, $B_4(f) = \frac{3t^2}{4} + \frac{t}{4}$. Make a conjecture about the formula for $B_n(f)$. Can you prove your conjecture?

Now, what is the use of these polynomials in equation 5? Here's an example to illustrate an application of them.

Question 11. Suppose that f is the function $f(x) = \dfrac{1}{1 + (x - 0.5)^2}$, $0 \le x \le 1$. Use equation 5 to show that

$$\begin{aligned} B_1(f) &= 0.8(1-t) + 0.8t (= 0.8) \\ B_2(f) &= 0.8(1-t)^2 + 2(1-t)t + 0.8t^2 \\ B_3(f) &= 0.8(1-t)^3 + 2.92(1-t)^2 t + 2.92(1-t)t^2 + 0.8t^3. \end{aligned}$$

When you graph the polynomials $B_1(f)$, $B_2(f)$, $B_3(f)$ and the function $f(x)$ in the same coordinate system, you get the picture in figure 5. The top graph is the graph of f and the graphs of $B_1(f)$, $B_2(f)$, $B_3(f)$ seem to get close to the graph of f.

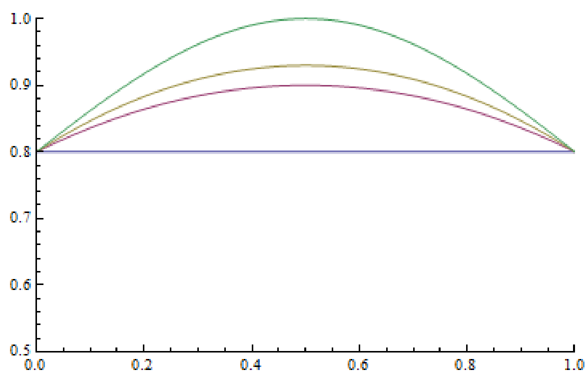

FIGURE 5. Polynomial Approximation $n = 3$

What do you think will happen when you calculate $B_4(f)$, $B_5(f)$, etc. In figure 6 below, I plot the graphs of the polynomial $B_n(f)$ for $n = 1, 2, \ldots, 50$ and the graph of $f(x) = \dfrac{1}{1 + (x - 0.5)^2}$ (the top curve) on the same coordinate system, as you can see, the larger n becomes, the closer the graph of $B_n(f)$ is to the graph of f.

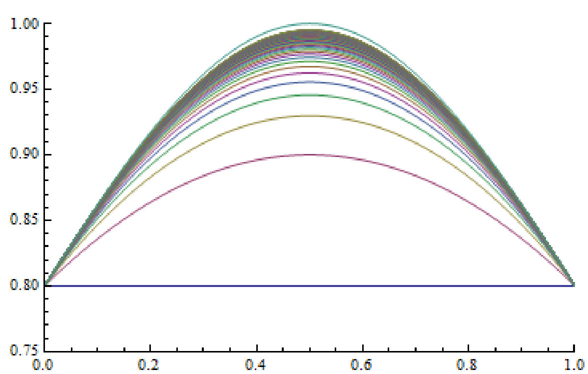

FIGURE 6. Polynomial Approximation $n = 50$

There is a deep and beautiful theorem in mathematics which says that if you have *any* continuous function defined on the interval $[0, 1]$, then the sequence of functions $B_n(f)$ as defined in equation 5 will get closer and closer to that function as n get larger and larger.

Question 12. Now, it is your turn. Choose a function f which is continuous on $[0, 1]$ that you like (other than x, x^2 and the one I have in question 11). Calculate $B_n(f)$ for $n = 1, 2, \ldots, 10$, or even a larger n if you'd like (and you don't have to do this by hand). Use any software or calculator of your choice to graph the $B_n(f)$, $n = 1, \ldots, 10$ and the funtion f in the same coordinate system. Print out the picture you get and submit it for this question.

Further research - optional - 5 extra points towards project points if answered. The polynomials $B_n(f)$ are not the only type of polynomial that can be used to approximate function. As a matter of fact, another type of polynomials that are also very useful in approximating functions is the class of Taylor polynomials, which you will study carefully in Cal II. Research Taylor polynomials and write a (roughly) 200 word essay to describe Taylor polynomials and their use.