HONORS CALCULUS I - PROJECT I - OF LIMITS, MONEY, e AND π

Abstract. In this project, we will consider the compound interest formula which is a standard formula in finance and economics and how it relates to one of the most important limits in Calculus. In the second part, we follow Archimedes' footsteps to derive the formula for the area of a circle.

1. Limits, e and interest rates

Simple interest: 5% simple interest on a loan of \$100 for 2 years means that 5% of the amount borrowed, the principal, is charged for every year of the loan. Therefore, the interest charged is $100 \times 0.05 \times 2 = \$10$ and the balance of the loan after 2 years is \$110.

Question 1. Show that the general formula for the balance A if an amount P borrowed or saved for t years with a simple interest rate r is

$$A = P(1 + rt).$$

Question 2. Banks rarely use simple interest. Explain why, preferably using a specific example to demonstrate the issue(s) with simple interest.

Normally, banks use *compound interest.* If you deposit \$100 into a saving account which pays 5% interest per year and is compounded semiannually (every six months) for 2 years, then your account balance after 2 years is calculated as follows. For the first 6 months (half a year), by the above formula for simple interest, the amount deposited will grow to $\$100(1 + 0.05 \times \frac{1}{2}) = \$102.5$. For the next 6 months, interest is calculated on the \$102.5 and it will grow to $\$102.5(1 + 0.05 \times \frac{1}{2}) = \$105.06$. After the next 6 months, the balance reaches \$107.69. After the final 6 months period, the balance will be \$110.38.

Question 3. In general, follow the idea in the above calculation to show that the balance A after an amount P being borrowed or saved at an interest rate r per year for t years, with compounding frequency n times per year, is given by the formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Question 4. Apply the above formula for compound interest to complete the following table which gives the balances after \$1000 is deposited in a saving account at 5% interest rate for 5 years with various compounding frequencies.

Compounding Frequency (n)	Balance (A)
Semiannually ($n = 2$)	
Quarterly ($n = 4$)	
Monthly ($n = 12$)	
Daily ($n = 365$)	
Hourly ($n = 8760$)	

You will notice that increasing the compounding frequencies more and more does not seem to noticeably change the final balance. So, if we continue this and compounded every minute, every second, every nanosecond, etc., it looks like we will reach some sort of limit. This limit is called continuous compounding. We will isolate the factor which gives rise to this behavior. First, let $m = \dfrac{n}{r}$, then the formula for compound interest becomes

$$A = P\left(1 + \frac{1}{m}\right)^{mrt} = P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}.$$

Note that as n becomes large, i.e., n approaches ∞, m will also approach ∞. Hence, it comes down to analyzing the limit

$$\lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^{m}.$$

Question 5. Evaluate the limit $\lim_{m\to\infty}\left(1+\dfrac{1}{m}\right)^m$ numerically by completing the following table (enter numerical results to 5 decimal places). What value (to 5 decimal places) to which $\left(1+\dfrac{1}{m}\right)^m$ seems to approach as $m\to\infty$?

m	$\left(1+\dfrac{1}{m}\right)^m$
10	
100	
1000	
10000	
100000	
1000000	

The limit you get for Question 5 is called e. If your calculation is correct, you should obtain $e \approx 2.71828$. Replace $\left(1+\frac{1}{m}\right)^m$ by e in the compound interest formula, we get the formula for *continuously compound interest*

$$A = Pe^{rt}.$$

However, there is an important issue we need to address here. How do we know that $\left(1+\frac{1}{m}\right)^m$ doesn't exceed 2.71828 or 3, or an even larger number if we continue to choose larger and larger values for m? Mathematically, the question is how do we know that the limit $\lim_{m\to\infty}\left(1+\dfrac{1}{m}\right)^m$ exists? In what follows, we will attempt to answer this question.

Question 6. Apply the Binomial Theorem (from Precalculus) to show that

$$\left(1+\frac{1}{m}\right)^m = 2+\frac{1}{2!}\left(1-\frac{1}{m}\right)+\frac{1}{3!}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)+\ldots+\left[\frac{1}{m!}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\cdots\frac{2}{m}\cdot\frac{1}{m}\right]$$

Now consider the sequence $\{x_m\}$ (recall what a sequence is from Precalculus) whose terms are defined by $x_m = \left(1+\dfrac{1}{m}\right)^m$. From the above, each term in the sum on the right hand side of x_m is a positive, increasing function of m and the number of terms increases as m increases. Therefore, the sequence $\{x_m\}$ is an *increasing sequence* (recall from Precalculus what an increasing sequence is).

Question 7. Explain why $x_m > 2$ for $m \geq 2$.

Now, in what follows, we will show that $x_m < 3$ for all m.

Question 8. Use the equation in Question 6 to explain why

$$x_m = \left(1+\frac{1}{m}\right)^m < 2+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{m!}$$

Question 9. Use the method of *mathematical induction* to show that

$$m! \geq 2^{m-1}, \text{ for } m \geq 1.$$

Therefore, from the result of Question 8 and Question 9, we have

$$\left(1+\frac{1}{m}\right)^m < 2+\left(\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{m-1}}\right).$$

Question 10. What kind of special sum is the sum inside the parentheses (recall Precalculus)? Determine the sum in terms of m.

Question 11. Use the result of Question 9 and Question 10 to explain why $x_m < 3$ for all m.

The result of Question 11 says that every term of the sequence x_m is smaller than 3, in other words, the sequence is *bounded above* by 3. Now, there is a mathematical theorem which says that if a sequence of numbers is increasing and bounded above, then that sequence must converge to a limit. The work above has demonstrated that the sequence $x_m = \left(1+\dfrac{1}{m}\right)^m$ is increasing and bounded above by 3. Therefore, it converges to a limit which is the number e. Moreover, e is between 2 and 3.

Question 12. Share your thoughts on the above reasoning which is an attempt to rigorously show that $\lim\limits_{m\to\infty}\left(1+\dfrac{1}{m}\right)^m$ exists and the limit is a number between 2 and 3. Is there anything that bothers you about the argument? Do you have any questions? Find out about at least one other application of the number e and write a short paragraph to explain the application.

2. Archimedes and the area of a circle

The formula $A = \pi R^2$ which gives the area of a circle with radius R is one of the most well-known formula in geometry. Have you ever wondered where this formula came from and how does the number π appear in the formula? In what follows, we will consider a method of deriving this formula, which originated with Archimedes, a leading Greek mathematician in the third century BCE.

First we cut the circle into n equal sectors where n is a positive integer. Each sector contains an isosceles triangle as in figure 1. Archimedes (and we) certainly know how to find the area of a triangle. And surely, summing the areas of these isosceles triangles gives an approximation to the area of the circle. More importantly, as n becomes larger and larger, these approximations seem to be closer and closer to the area of the circle. What would happen if n approaches infinity?

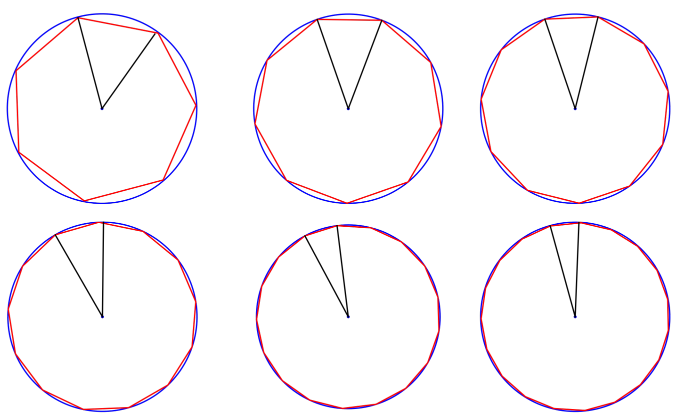

Figure 1. Divide and conquer

Question 13. Consider the isosceles triangle in figure 2 below where θ is the vertex angle and R is the radius of the circle. Show that the area of the triangle is

$$\text{Area } = \frac{1}{2}R^2\sin\theta.$$

(Hint: draw the height from the vertex at the origin, express the height and base in terms of R and θ and apply the double angle formula $\sin(2x) = \frac{1}{2}\sin(x)\cos(x)$.)

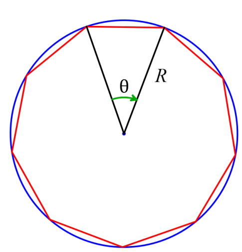

Figure 2. A little triangle

Question 14. If there are n isosceles triangles, find a relationship between n and θ and solve for θ in terms of n. Then demonstrate that the sum of the areas of the n isosceles triangles is

$$\text{Sum of areas of triangles } = \frac{1}{2}nR^2\sin\left(\frac{2\pi}{n}\right).$$

Question 15. Now find out what happens to the sum of the areas of the n triangles when n approaches infinity by finding the limit

$$\lim_{n\to\infty}\frac{1}{2}nR^2\sin\left(\frac{2\pi}{n}\right).$$

Note: I would like for you to find the above limit analytically, i.e., do not find the limit by plugging in larger and larger values of n into the expression. (Hint: you will need the important limit that we study in class $\lim_{x\to 0}\frac{\sin x}{x} = 1$, try to relate this limit to the limit you are finding.)

Your answer should agree with your expectation of seeing the formula for the area of the circle. However, we must make sure that the above process does not miss any of the area. In what follows, we will estimate how far off our approximation of the area of the circle using these isosceles triangles is and show that this error approaches 0 as n, the number of triangles, approaches infinity.

At the base of the isosceles triangle in a sector, draw the smallest rectangle that contains the area of the sector outside of the triangle as in figure 3 below.

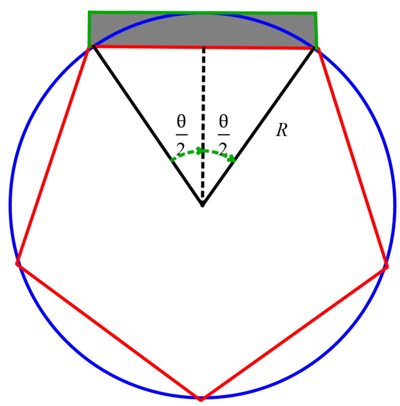

FIGURE 3. Error estimate

Question 16. Show that the area of the shaded rectangle in figure 3 is given by the formula

$$\text{Area of shaded rectangle } = 2R^2\sin\left(\frac{\theta}{2}\right)\left(1-\cos\left(\frac{\theta}{2}\right)\right)$$

Note that when we sum up all these shaded areas, we get an upper bound for the aforementioned error.

Question 17. Express θ in terms of n again and show that an upper bound for the total error when we use n triangles to approximate the area of the circle is $2R^2n\sin\left(\frac{\pi}{n}\right)\left(1-\cos\left(\frac{\pi}{n}\right)\right)$. Then show that as n approaches infinity, this upper bound, and hence the error, approaches 0.

Question 18. What is the key idea of the above technique that we just go through to derive the formula for the area of the circle? Do you think we can find or approximate areas or volumes of other geometric shapes using Archimedes' technique? Choose a geometric shape and write a short paragraph to describe how you would derive the formula for its area or volume from scratch if you were Archimedes, for example, how would you approximate the volume of a sphere in Archimedes' way?