3.7. Derivatives of Inverse Functions

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Goal: ① Review of Inverse Functions / Inverse Trig Functions

② The Inverse Function Theorem

③ Derivatives of Inverse Trig Functions.

Review:

$$\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$$

$$\arcsin(-3) \quad \text{DNE}$$

To sum up,

$\arcsin(x)$ gives you an angle y in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is equal to x, i.e.,

$$\sin(y) = x$$

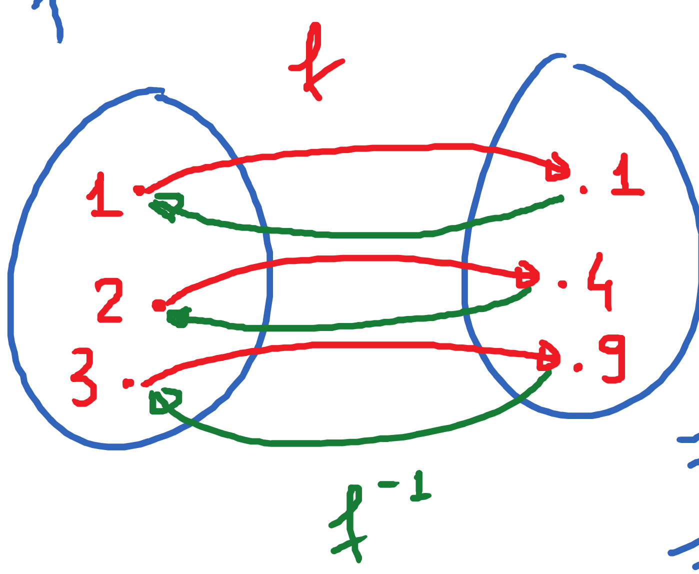
arccos(x) gives you an angle y in $[0,\pi]$ such that $\cos(y) = x$.

arctan(x) gives you an angle y in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan(y) = x$.

etc.

In general, if $y = f(x)$ is a one-to-one function on a domain, then the inverse function of y, denoted by $f^{-1}(x)$ (not the same as $\frac{1}{f(x)}$) is the function that undoes what f does.

E.g.



$f(1) = 1$

$f(2) = 4$

$f(3) = 9$

$f^{-1}(1) = 1$

$f^{-1}(9) = 3$

$f^{-1}(4) = 2$

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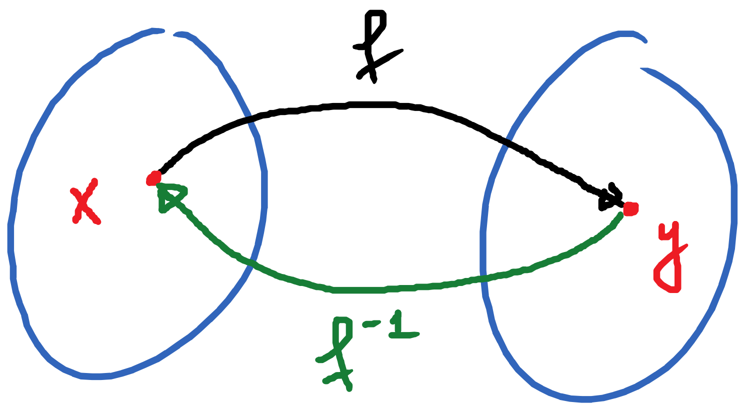
To sum up,

$$\boxed{\begin{aligned} f^{-1}(f(x)) &= x \\ f(f^{-1}(x)) &= x. \end{aligned}}$$



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Inverse Function Theorem

Let f be a function that is differentiable and invertible on an interval containing x.

Then:

$$\boxed{\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'\left(f^{-1}(x)\right)}}$$

E.g. Given that $f(4) = 5$; $f^{-1}(5) = 4$
and $f'(4) = 2$.

Find the derivative of f^{-1} at $x = 5$; i.e.,

$$\left.\frac{d}{dx}\left(f^{-1}(x)\right)\right|_{x=5} = \frac{1}{f'\left(\boxed{f^{-1}(5)}\right)} \rightarrow 4$$

$$= \frac{1}{f'(4)} = \boxed{\frac{1}{2}}$$

E.g. Given $f(x) = x^5 + 3x^3 - 4x - 8$

$f(1) = -8$; $f^{-1}(-8) = 1$

ⓐ Find the derivative of f^{-1} at $x = -8$.

$$\left.\frac{d}{dx}\left(f^{-1}(x)\right)\right|_{x=-8} = \frac{1}{f'\left(\boxed{f^{-1}(-8)}\right)}$$

$$= \frac{1}{\boxed{f'(1)}}$$

$f'(x) = 5x^4 + 9x^2 - 4$

$f'(1) = 5 + 9 - 4 = 10$

So, $(f^{-1})'(-8) = \boxed{\frac{1}{10}}$

(b) Find the equation of the tangent line to the graph of f^{-1} at $(-8, 1)$.

Slope = derivative of f^{-1} at -8 = $\frac{1}{10}$.

$$\rightarrow \quad y - 1 = \frac{1}{10}(x+8)$$

$$\boxed{y = \frac{1}{10}x + \frac{9}{5}}$$

E.g. Given $f(x) = x^3 + 2x + 3$.

$f(-1) = 0$

Find the equation of the tangent line to the graph of f^{-1} at the point $(0, -1)$.

Sol: First Step: Slope $= (f^{-1})'(0)$

$\rightarrow$ we need to find $(f^{-1})'(0)$

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$$(f^{-1})'(0) \overset{\text{IFT}}{=} \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(-1)}$$

$$f'(x) = 3x^2 + 2 \longrightarrow f'(-1) = 3(-1)^2 + 2 = 5.$$

$$(f^{-1})'(0) = \frac{1}{5}.$$

Tangent line : $y - (-1) = \frac{1}{5}x$

$$\boxed{y = \frac{1}{5}x - 1}$$

Why is the I.F.T true ?

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

chain rule

Key: We know $\boxed{f(f^{-1}(x)) = x}$

Take the derivative w.r.t. x of both sides :

$$\underbrace{f'(f^{-1}(x))}_{\text{der. outside}} \cdot \underbrace{(f^{-1})'(x)}_{\text{der. inside}} = 1 \longrightarrow (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$