4.2. Linear Approximation and Differentials

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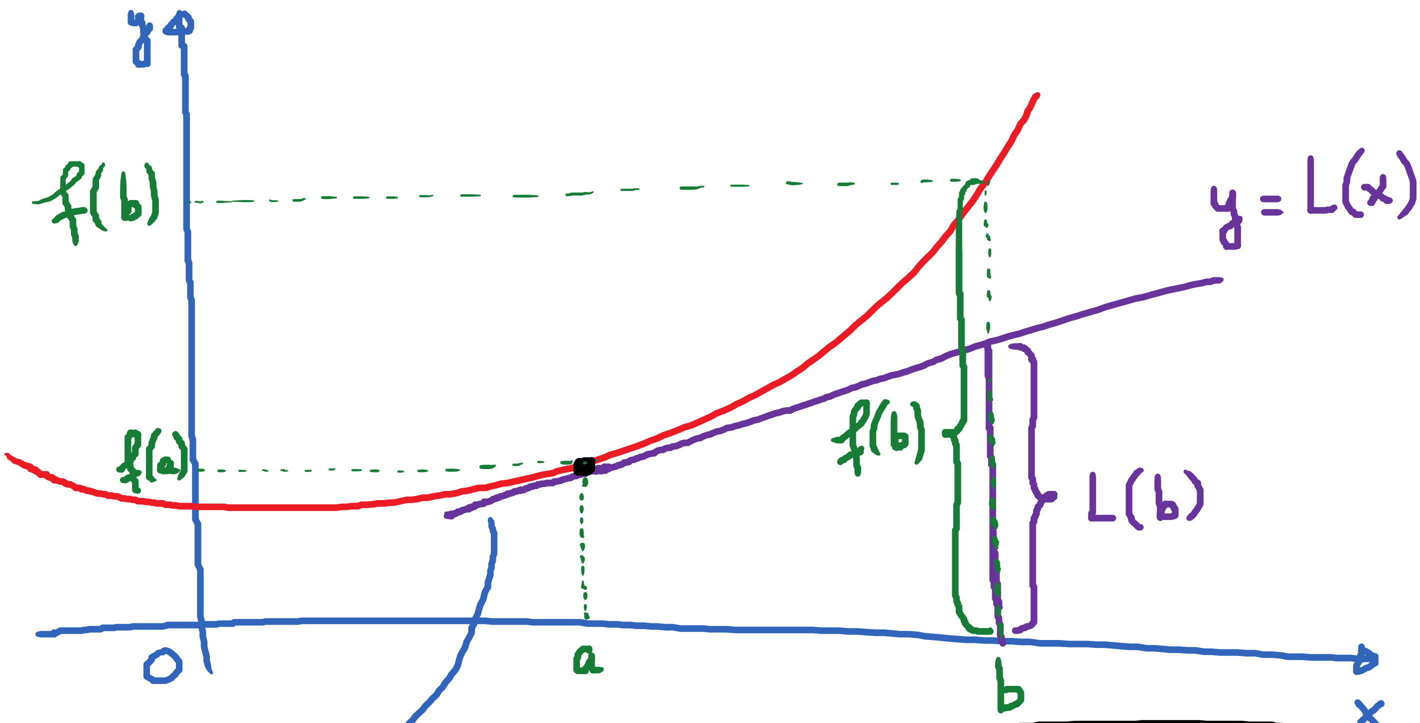
Goals: (1) Understand and Apply the formula for the linear approximation of a function

(2) Understand the concept of the differential of a function

(1) Linear Approximation.



tangent line to the graph of f at $(a, f(a))$

$$\boxed{L(b) \approx f(b)}$$

$L(b)$ is a linear approximation to $f(b)$

Goal: Find the formula for $y = L(x)$; i.e., find the equation of the tangent line to the graph of $y = f(x)$ at $(a, f(a))$

Slope $= f'(a)$

Point of tangency $= (a, f(a))$

Point-Slope equation of the tangent line:

$$y - f(a) = f'(a)(x-a)$$

$$\longrightarrow y = f(a) + f'(a)(x-a)$$

$$\longrightarrow \boxed{L(x) = f(a) + f'(a)(x-a)}$$

$\longrightarrow$ The formula for the linear approximation of the function $y = f(x)$ at $x = a$ is:

$$\boxed{L(x) = f(a) + f'(a)(x-a)}$$

E.g. let $f(x) = \sqrt{x}$

(a) Find the linear approximation $y = L(x)$ to f at $a = 9$.

(b) Use this linear approximation $y = L(x)$ to approximate $\sqrt{9.1}$

(a) $$L(x) = f(9) + f'(9)(x-9)$$

$$f(9) = \sqrt{9} = 3$$

$$f'(x) = \frac{1}{2\sqrt{x}} \longrightarrow f'(9) = \frac{1}{2 \cdot \sqrt{9}} = \frac{1}{6}.$$

$$L(x) = 3 + \frac{1}{6}(x-9)$$

(b) The approximation for $\sqrt{9.1}$ using this is :

$$L(9.1) = 3 + \frac{1}{6}(\underbrace{9.1 - 9}_{0.1}) = 3 + \frac{0.1}{6}$$

$$= \boxed{3.0167} \longrightarrow$$ linear approximation to $f(x) = \sqrt{9.1}$ at $x = 9.1$

Ex. Use a linear approximation to estimate

$\sqrt[3]{1001}$.

(You need to figure out the function f, the point a to write the equation of $L(x)$.)

Function to use: $f(x) = \sqrt[3]{x} = x^{1/3}$

$a = 1000$.

$L(x) = f(a) + f'(a)(x-a)$

$f(a) = \sqrt[3]{1000} = 10$

$f'(x) = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$

$f'(x) = \frac{1}{3(\sqrt[3]{x})^2}$; $f'(a) = f'(1000) = \frac{1}{3(\sqrt[3]{1000})^2}$

$= \frac{1}{300}$

$$\boxed{L(x) = 10 + \frac{1}{300}(x-a)}$$

$\longrightarrow$ formula for linear approx. near $a = 1000$

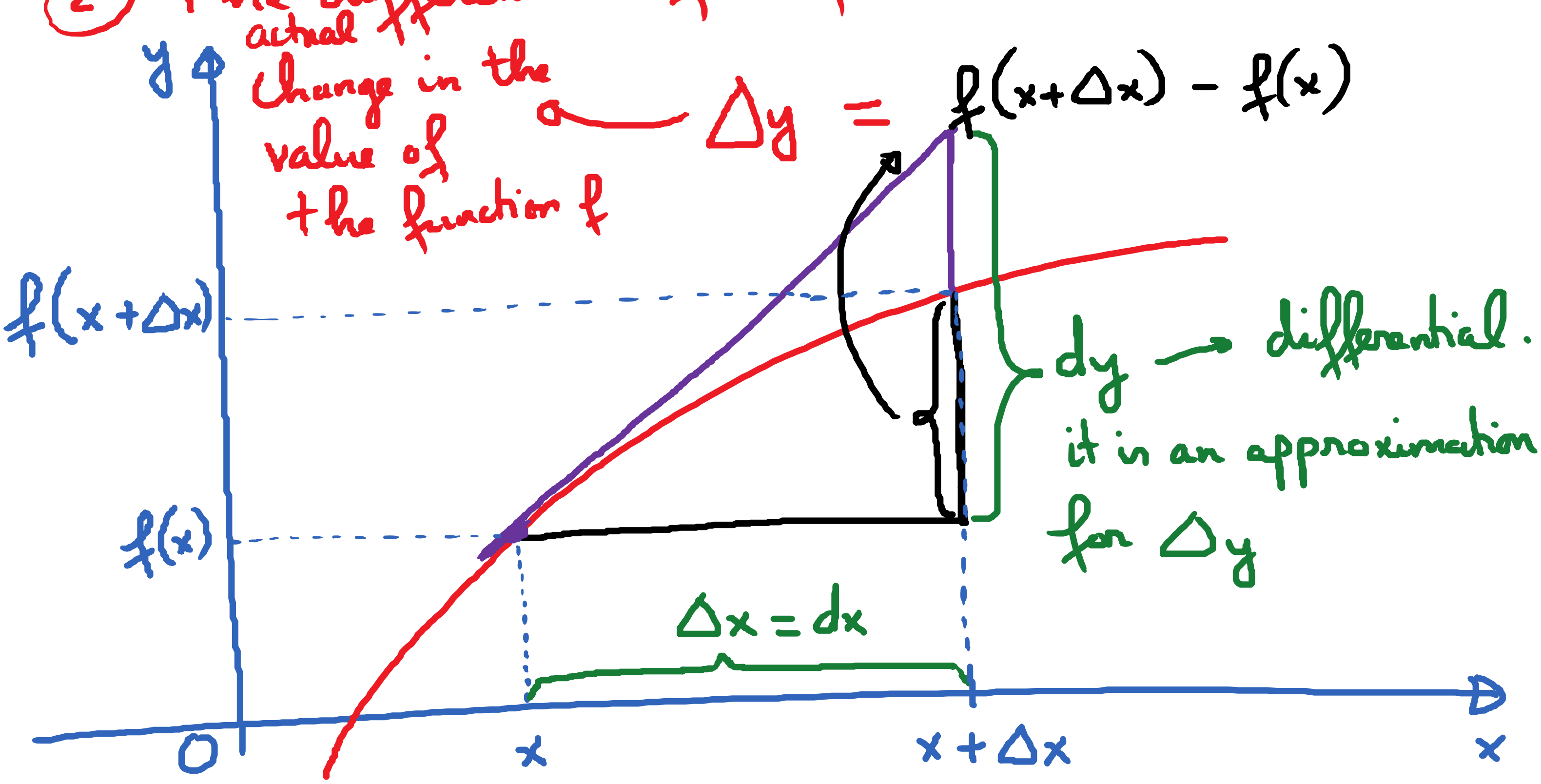
To approximate $\sqrt[3]{1001}$, we plug 1001 into $L(x)$

$$L(1001) = 10 + \frac{1}{300}(1001 - 1000)$$

$$= 10 + \frac{1}{300} \cdot 1 = 10 + \frac{1}{300}$$

$$\approx 10.00333\ldots$$

② The differential of a function.



$$\boxed{dy \approx \Delta y}$$

$$\frac{dy}{dx} = \frac{\text{Rise}}{\text{Run}} = \text{Slope of the tangent line} = f'(x)$$

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$$\frac{dy}{dx} = f'(x) \longrightarrow \boxed{dy = f'(x) \cdot dx}$$

This is the formula for the differential of the function $y = f(x)$ at a point.

dy can be used to approximate the actual change Δy of the function when x changes to $x + \Delta x$.

E.g. $y = f(x) = x^3 + x^2 - 2x + 1$.

Find Δy and find dy as x changes from 2 to 2.05

Δy = actual change in the function value

$= f(x + \Delta x) - f(x)$

$\Delta y = f(2.05) - f(2) \approx \boxed{0.71763}$

$$\boxed{dy = f'(x)\,dx}$$

$$dx = \Delta x = 2.05 - 2 = 0.05$$

$$f'(x) = 3x^2 + 2x - 2 \rightarrow f'(2) = 3\cdot(2)^2 + 2\cdot(2) - 2$$
$$= 14$$

$$\rightarrow dy = 14\cdot(0.05) = \boxed{0.7}$$ → an approximation to Δy.

differential.

#16 (Review sheet)

$$V = x^3$$

$$\longrightarrow dV = 3x^2\,dx$$

$$dx = 11.1 - 11 = 0.1 \;;\quad x = 11$$

$$\rightarrow dV = 3\cdot(11)^2\cdot(0.1)$$
$$= (363)\cdot(0.1) = \boxed{36.3}$$