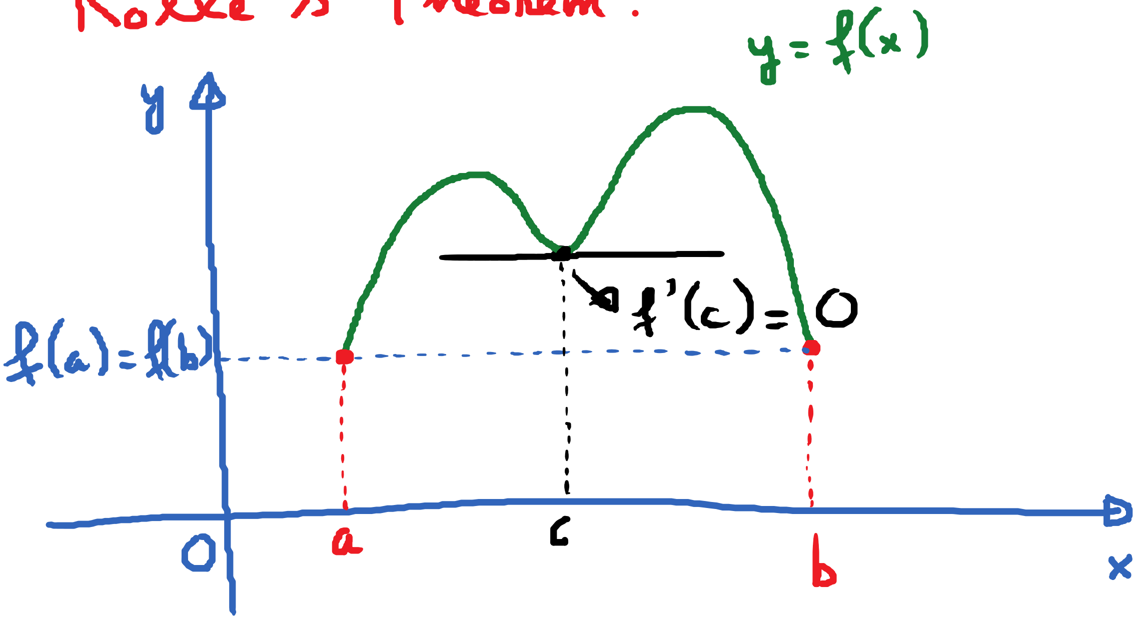
4.4. Rolle's Theorem and the Mean Value Theorem

Tuesday, July 31, 2018 10:04 AM

Rolle's Theorem.



Rolle's Theorem

f : function on an interval $[a,b]$

1. f is continuous on $[a,b]$
2. f is differentiable on (a,b)
3. $f(a) = f(b)$

Hypothesis of Rolle's Theorem

$\Longrightarrow$ Conclusion: There exists a number c in the interval (a,b) such that $f'(c) = 0$

E.g. $f(x) = x^3 - 4x$; on $[-2,2]$

ⓐ Verify that f satisfies all the conditions of Rolle's Theorem.

ⓑ Find the value(s) of c that satisfy the conclusion of the theorem.

Solution

ⓐ Condition 1: Is f continuous on $[-2,2]$? ✓

Yes. Because f is a polynomial.

Condition 2: Is f differentiable on $[-2,2]$? ✓

Yes. $f'(x) = 3x^2 - 4$. It exists everywhere on $[-2,2]$.

Condition 3: Is $f(2) = f(-2)$? ✓

$f(2) = 0$; $f(-2) = 0$

Yes.

Conclusion: There exists a number c in $(-2,2)$ s.t. $f'(c)=0$

(b) To find c, we solve: $f'(c)=0$

$f'(x) = 3x^2 - 4.$

So, $f'(c)=0 \longleftrightarrow 3c^2-4=0$

$\longleftrightarrow 3c^2 = 4$

$\longleftrightarrow c^2 = \frac{4}{3}$

$\longleftrightarrow c = \pm\sqrt{\frac{4}{3}} = \pm\frac{2}{\sqrt{3}}$

E.g. $f(x) = \sqrt{x}\cdot(8-x)$ on $[0,8]$

$= 8\sqrt{x} - x^{3/2} \longrightarrow$ continuous on $[0,8]$

$f'(x) = \frac{4}{\sqrt{x}} - \frac{3}{2}x^{1/2} = \frac{4}{\sqrt{x}} - \frac{3\sqrt{x}}{2}$

$f'(x) = \frac{8-3x}{2\sqrt{x}}$ $\longrightarrow$ exists on $(0,8)$
$\longrightarrow$ diff. on $(0,8)$

$f(0) = 0 = f(8)$

$\Longrightarrow$ Hypothesis of Rolle's Theorem is satisfied

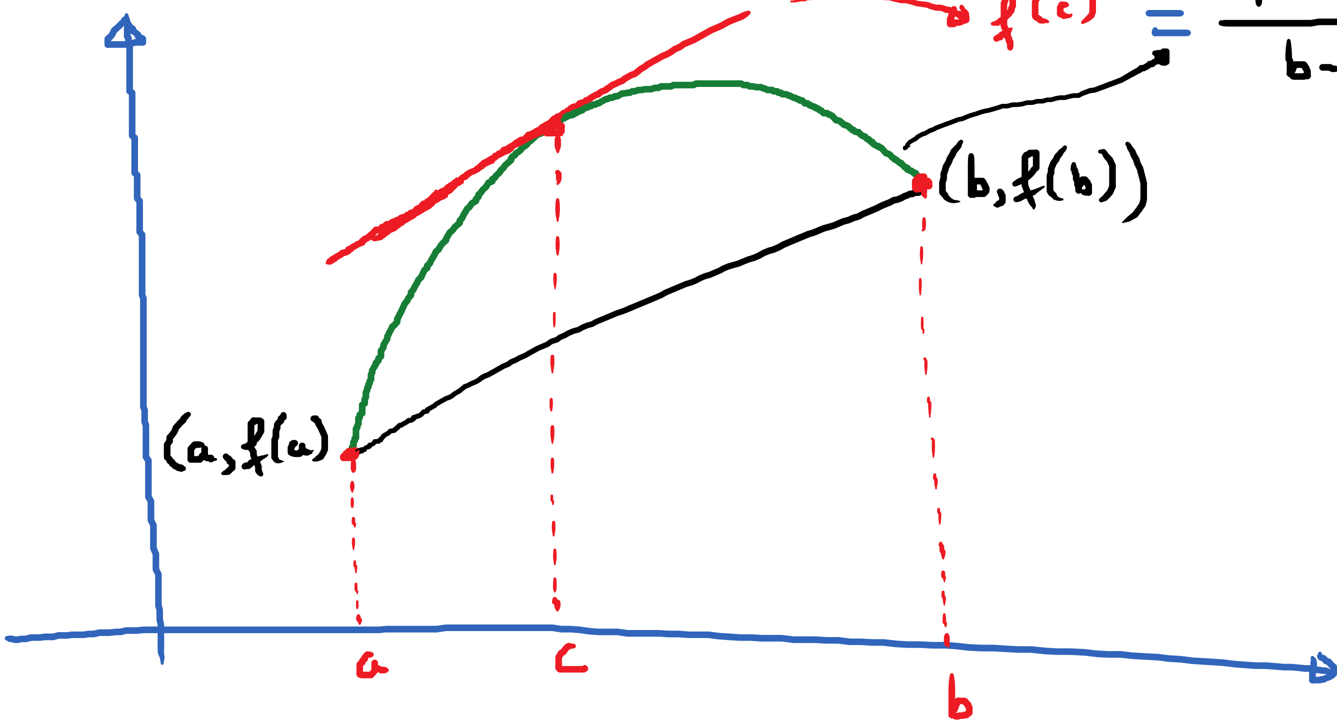
Solve $f'(c) = 0$

$$\frac{8-3c}{\sqrt{c}} = 0 \iff 8 - 3c = 0$$

$$\iff \boxed{c = \frac{8}{3}}$$

The Mean Value Theorem (M.V.T)

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



MVT

f: function on $[a,b]$

1. f is continuous on $[a,b]$
2. f is differentiable on (a,b)

$\}$ Hypothesis of MVT

$\Longrightarrow$ Conclusion: There exists a number c in (a,b) such that:

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$

E.g. $f(x) = \ln(x)$ on $[1,4]$

a. Verify that the conditions of the MVT are satisfied.

b. Find the value(s) c in the conclusion of the theorem.

(1) Is f continuous on $[1,4]$? ✓

$f(x) = \ln(x)$ is continuous on its domain $(0,\infty)$.

$[1,4]$ is contained in this domain.

(2) Is f differentiable on $(1,4)$? ✓

$f'(x) = \frac{1}{x} \to$ it exists on $(1,4)$

Hypothesis of MVT is satisfied.

$\Longrightarrow$ there exists c s.t. $f'(c) = \frac{f(b) - f(a)}{b - a}$.

(b) Find c ? $f(x) = \ln(x)$

$f'(x) = \frac{1}{x}$; $a = 1$; $b = 4$

$$\frac{1}{c} = \frac{\ln(4) - \cancel{\ln(1)}}{4 - 1} = \frac{\ln(4)}{3}$$

0

$$\frac{1}{c} = \frac{\ln(4)}{3} \to \boxed{c = \frac{3}{\ln(4)}}$$