4.6. Limits at Infinity and Asymptotes.

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Goals: ① Find limits at infinity

② Find vertical asymptotes and horizontal asymptotes of functions.

Basic limits at Infinity.

① $f(x) = x^n$; n is a positive integer.

$f(x) = x^2$; $f(x) = x^3$; $f(x) = x^{2018}$

$$\lim_{x \to \infty} [x^n] = \infty$$

$$\lim_{x \to -\infty} [x^n] = \begin{cases} \infty & \text{if } n \text{ is even} \\ -\infty & \text{if } n \text{ is odd} \end{cases}$$

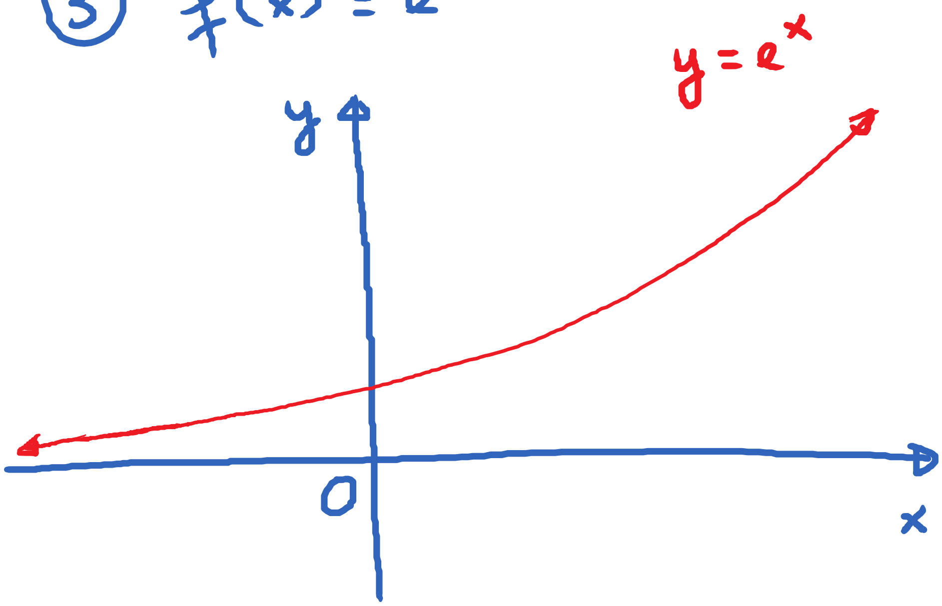
② $f(x) = \dfrac{1}{x^n}$; n is a positive integer

$$\left.\begin{array}{l} \lim\limits_{x \to \infty} \left[\dfrac{1}{x^n} \right] = 0 \\ \\ \lim\limits_{x \to -\infty} \left[\dfrac{1}{x^n} \right] = 0 \end{array}\right\} \lim\limits_{x \to \pm\infty} \left[\dfrac{1}{x^n} \right] = 0$$

③ $f(x) = e^x$

$$\lim_{x \to \infty} \left[e^x \right] = \infty$$

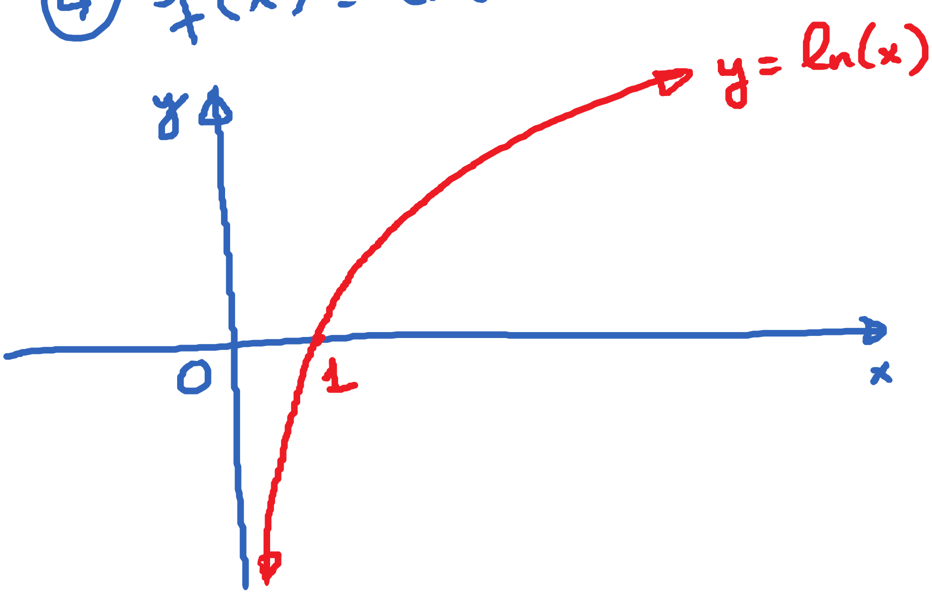
$$\lim_{x \to -\infty} \left[e^x \right] = 0$$



④ $f(x) = \ln(x)$

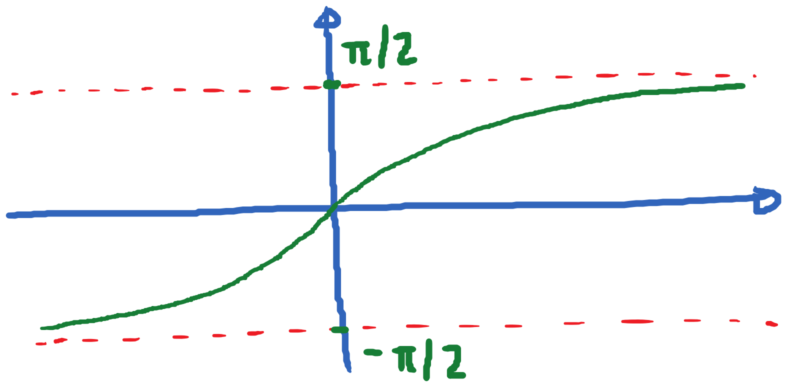
$$\lim_{x \to \infty} \left[\ln(x) \right] = \infty$$

$$\lim_{x \to 0^+} \left[\ln(x) \right] = -\infty$$



5 $f(x) = \arctan(x)$

$$\lim_{x\to\infty} \left[\arctan(x)\right] = \frac{\pi}{2}$$



$$\lim_{x\to-\infty} \left[\arctan(x)\right] = -\frac{\pi}{2}.$$

E.g. of finding limits at infinity of functions.

E.g.

$$\lim_{x\to\infty} \frac{\boxed{x^2} + 2x + 7}{3x^2 + 5} \approx \frac{\cancel{x^2}}{3\cancel{x^2}} = \boxed{\frac{1}{3}}$$

$$\lim_{x\to\infty} \frac{x^2 + 2x + 7}{3x^4 + 17} \approx \frac{x^2}{3x^4} = \frac{1}{3x^2} = \boxed{0}$$

$$\lim_{x\to\infty} \frac{x^4 + 2x + 7}{3x^2 + 17} \approx \frac{x^4}{3x^2} = \frac{x^2}{3} = \boxed{\infty}$$

$$\lim_{x\to-\infty} \frac{x^5 + 2x + 7}{3x^2 + 17} \approx \frac{x^5}{3x^2} = \frac{x^3}{3} = \boxed{-\infty}$$

$$\lim_{x\to-\infty} \frac{x^7 + 2x + 7}{x^3 + 17} \approx \frac{x^7}{x^3} = x^4 = \boxed{\infty}$$

$$\lim_{x\to\infty}\frac{3x+2}{\sqrt{4x^2+5}} = \lim_{x\to\infty}\frac{3x}{\sqrt{4x^2}} = \lim_{x\to\infty}\frac{3\cancel{x}}{2\cancel{x}}$$

$$= \frac{3}{2}$$

$$\lim_{x\to-\infty}\frac{3x+2}{\sqrt{4x^2+5}} = \lim_{x\to-\infty}\frac{3x}{\sqrt{4x^2}} = \lim_{x\to-\infty}\frac{3\cancel{x}}{-(2\cancel{x})}$$

$$= \lim_{x\to-\infty}\left(-\frac{3}{2}\right) = \boxed{-\frac{3}{2}}$$

$$\sqrt{A^2} = \begin{cases} A & \text{if } A>0 \\ -A & \text{if } A<0 \end{cases}$$

$$\lim_{x\to\infty}\frac{3x^2+2}{\sqrt{4x^2+5}} = \lim_{x\to\infty}\frac{3x^2}{\sqrt{4x^2}} = \lim_{x\to\infty}\frac{3x^{\cancel{2}}}{2\cancel{x}} = \boxed{\infty}$$

$$\lim_{x\to-\infty}\frac{3x^2+2}{\sqrt{4x^2+5}} = \lim_{x\to-\infty}\frac{3x^2}{\sqrt{4x^2}} = \lim_{x\to-\infty}\frac{3x^2}{(-2x)}$$

$$= \lim_{x\to-\infty}\frac{3x}{-2} = \boxed{\infty}$$

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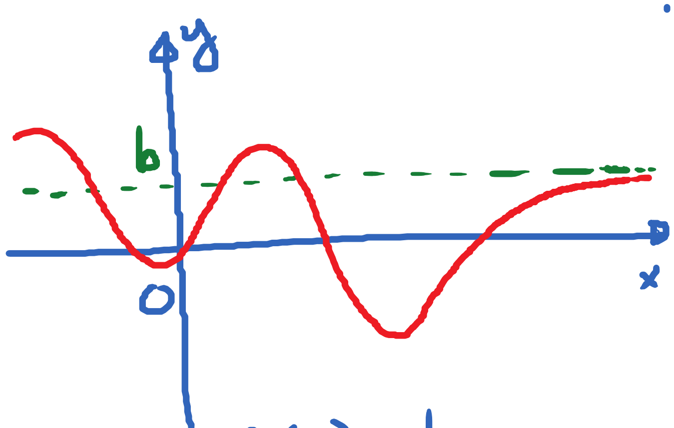
E.g. $\lim_{x\to\infty} \frac{1-e^x}{1+2e^x} = \lim_{x\to\infty} \frac{-e^x}{2e^x} = -\frac{1}{2}$

$\lim_{x\to\infty} \frac{1-e^{2x}}{1+2e^x} = \lim_{x\to\infty} \frac{-e^{2x}}{2e^x} = \lim_{x\to\infty} (-e^x)$

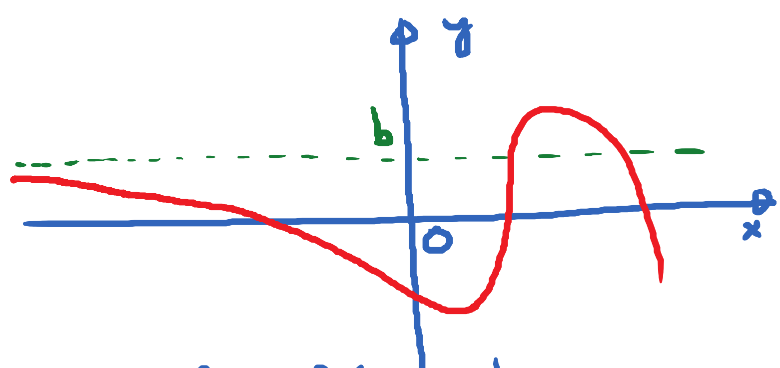
$= \boxed{-\infty}$.

$\lim_{x\to\infty} \arctan(\underbrace{\boxed{e^x}}_{\to\infty}) = \frac{\pi}{2}$

Def: The horizontal line $y=b$ is a horizontal asymptote of the function $y=f(x)$ if either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$



$\lim_{x\to\infty} f(x) = b$



$\lim_{x\to-\infty} f(x) = b$

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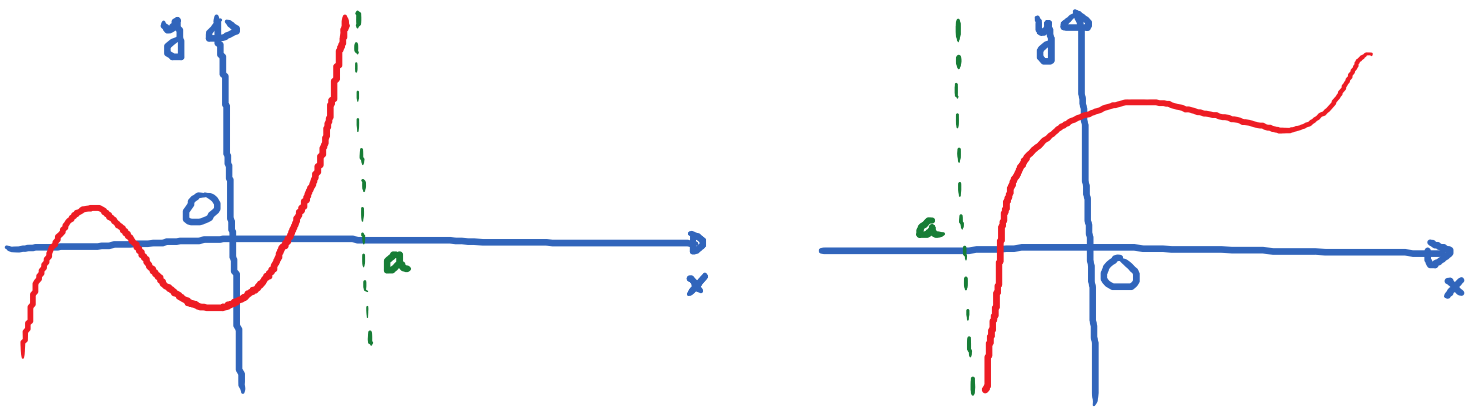
Def: If $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$

then the vertical line $x = a$ is a vertical asymptote of the function $y = f(x)$



$\lim_{x \to a^-} f(x) = \infty$ $\lim_{x \to a^+} f(x) = -\infty$

Strategy for finding V.A. of rational functions

$$y = \frac{p(x)}{q(x)}.$$

1. Factor $p(x)$ and $q(x)$ completely.
2. Cancel the common factor(s) (if any)
3. Values of x for which the denominator of the simplified expression is zero give rise to V.A.

E.g. $f(x) = \dfrac{x^2 - 9}{x^2 - 4x + 3}$

① $f(x) = \dfrac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}(x-1)}$

② Cancel

③ Simplified expression : $\dfrac{x+3}{x-1}$

$\longrightarrow$ V.A. $\boxed{x = 1}$.