4.9. Newton's Method

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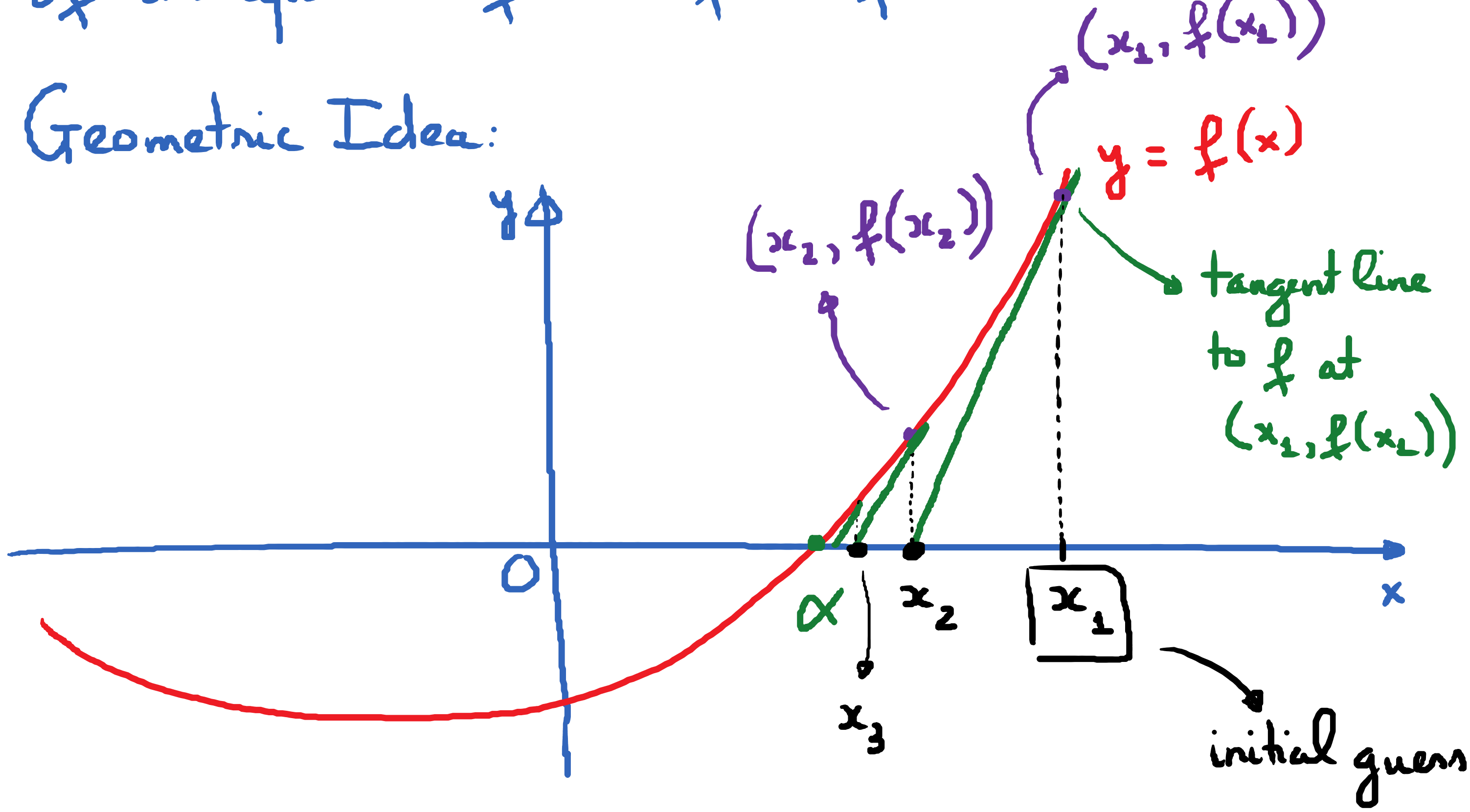
Goal: Apply Newton's method to estimate solutions to an equation.

Newton's method is used to estimate a solution of an equation of the form $f(x) = 0$

Geometric Idea:



α is a solution to the equation $f(x) = 0$
(it is the x-coordinate of a x-intercept of f)

Implementation of the method

Step 1: Start with an initial guess x_1.

Step 2: x_2 = intersection between the tangent line to the graph of $y = f(x)$ at $(x_1, f(x_1))$ and the x-axis.

x_2 is the second approximation to α.

As in picture, x_2 is closer to α than x_1

Step 3: Repeat the process to obtain $x_3, x_4, x_5, \dots$

The sequence $x_1, x_2, x_3, x_4, x_5, \dots$ are better and better approximation to α.

Goal now: Develop a formula to find x_2 given x_1, find x_3 from x_2, find x_4 from x_3, etc.

Formula to find x_2.

Step 1: Equation of the tangent line to the graph of $y = f(x)$ at $(x_1, f(x_1))$

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Slope: $f'(x_1)$. Point $(x_1, f(x_1))$

Point - Slope Equation:

$$y - f(x_1) = f'(x_1)(x - x_1)$$

$\longrightarrow$ Slope - Intercept Equation:

$$y = f'(x_1)(x - x_1) + f(x_1)$$

Step 2: To find x_2, we set $y = 0$ in the above equation:

$$0 = f'(x_1)(x - x_1) + f(x_1)$$

$$-f(x_1) = f'(x_1)(x - x_1)$$

$$\frac{-f(x_1)}{f'(x_1)} = x - x_1$$

$$x = \boxed{x_1 - \frac{f(x_1)}{f'(x_1)}}$$

this gives us x_2

So, $$\boxed{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}}$$

Similarly,

$$\boxed{x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}}$$

In general, we have :

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 1}$$

This is the formula for Newton's method which gives us the $(n+1)$-approximation x_{n+1} from the $n^{\underline{th}}$ approximation x_n.

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E.g. Use Newton's method with initial guess $x_1 = 2$ to estimate the solution (root, zero) of the equation: $\underbrace{x^3 + 5x - 1}_{f(x)} = 0$

$f(x) \longrightarrow f'(x) = 3x^2 + 5$

Find x_2, x_3, x_4.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{(2)^3 + 5(2) - 1}{3(2)^2 + 5}$$

$$\boxed{x_2 = 1}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1 - \frac{f(1)}{f'(1)} = \frac{3}{8} \approx \boxed{0.375}$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = \frac{3}{8} - \frac{f(\frac{3}{8})}{f'(\frac{3}{8})} \approx \boxed{0.204}$$

Ex. Use Newton's method to estimate $\sqrt{2}$.

The number $\sqrt{2}$ is the positive solution to the equation

$$\underbrace{x^2 - 2}_{f(x)} = 0$$

$f(x) \longrightarrow f'(x) = 2x$

① Find the formula that relates x_{n+1} to x_n in Newton method for this function.

② Start with $x_1 = 3$, find x_2, x_3.

① $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{x_n \cdot 2x_n}{1 \cdot 2x_n} - \frac{x_n^2 - 2}{2x_n}$$

$$x_{n+1} = \frac{2x_n^2 - (x_n^2 - 2)}{2x_n} = \frac{x_n^2 + 2}{2x_n}$$

$$\boxed{x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)}$$

② $x_2 = \frac{11}{6} \simeq 1.833\ldots$; $x_3 \simeq 1.462\ldots$