5.1 and 5.2 - Areas and Definite Integrals

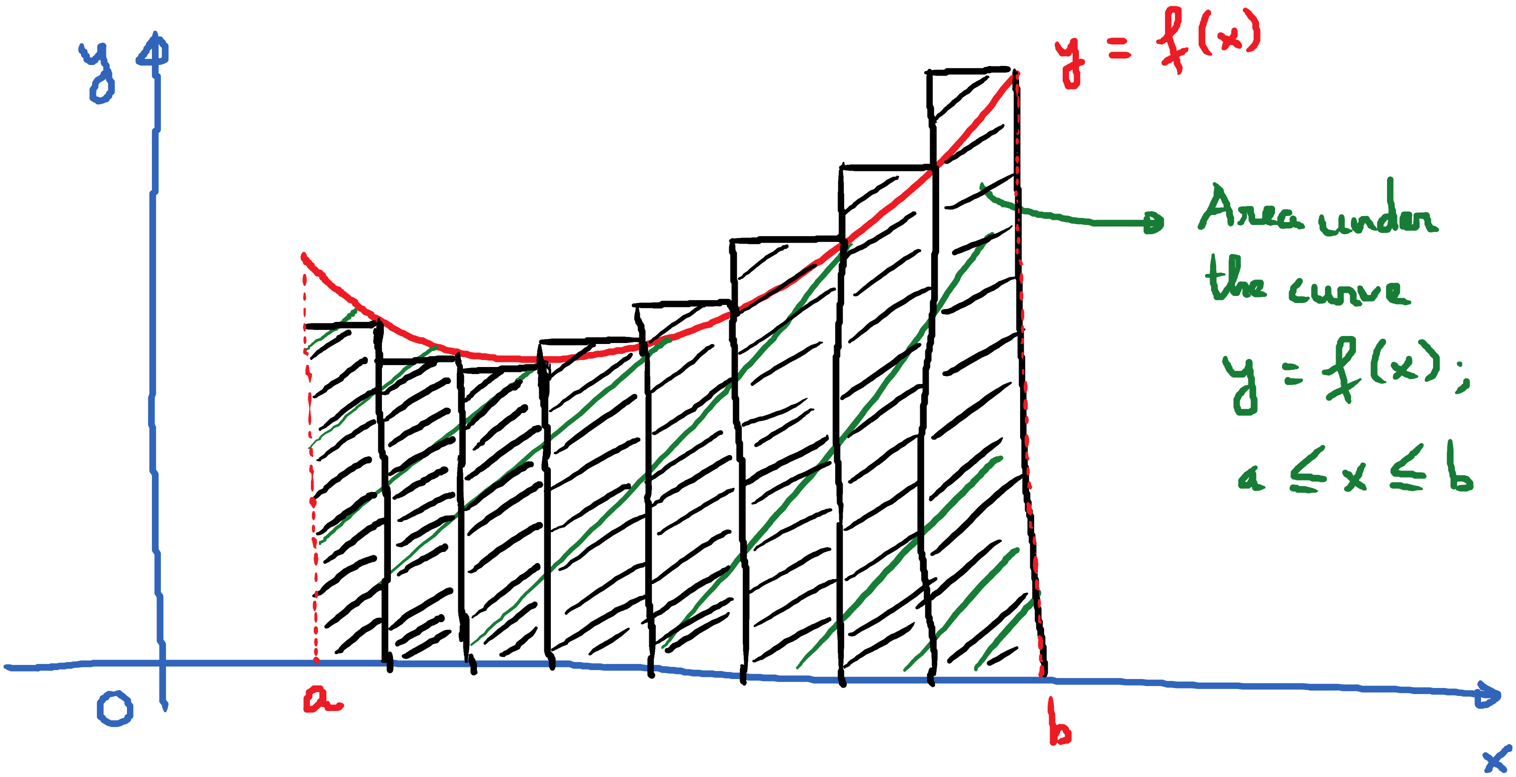
Wednesday, August 8, 2018 7:34 AM

<u>Recall:</u> Antiderivatives (Indefinite Integrals)

$\int \boxed{f(x)}\,dx = F(x) + C$ where $F(x)$ is a function whose derivative equals $f(x)$

Look for a function whose derivative is $f(x)$

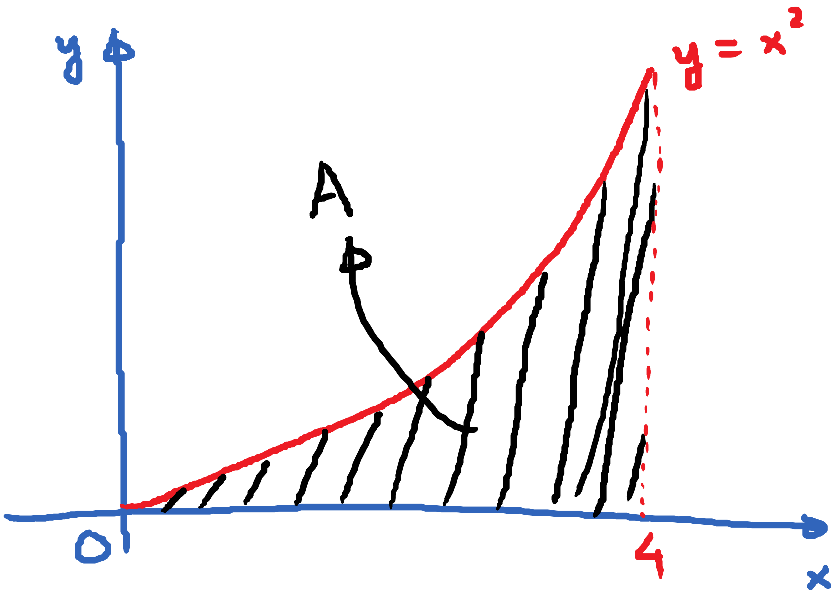
→ We will develop the concept of the definite integral in this section.



Using Left Sums or Right Sums to Approximate Areas

Let $f(x) = x^2$.

Consider this function on $[0,4]$.



Q: Approximate the area under the curve $y = x^2$ on $[0,4]$ using L_4 and R_4

L_4 = Left endpoints approximation (Left Riemann Sum with 4 subintervals)

R_4 = Right endpoints approximation (Right Riemann Sum with 4 subintervals)

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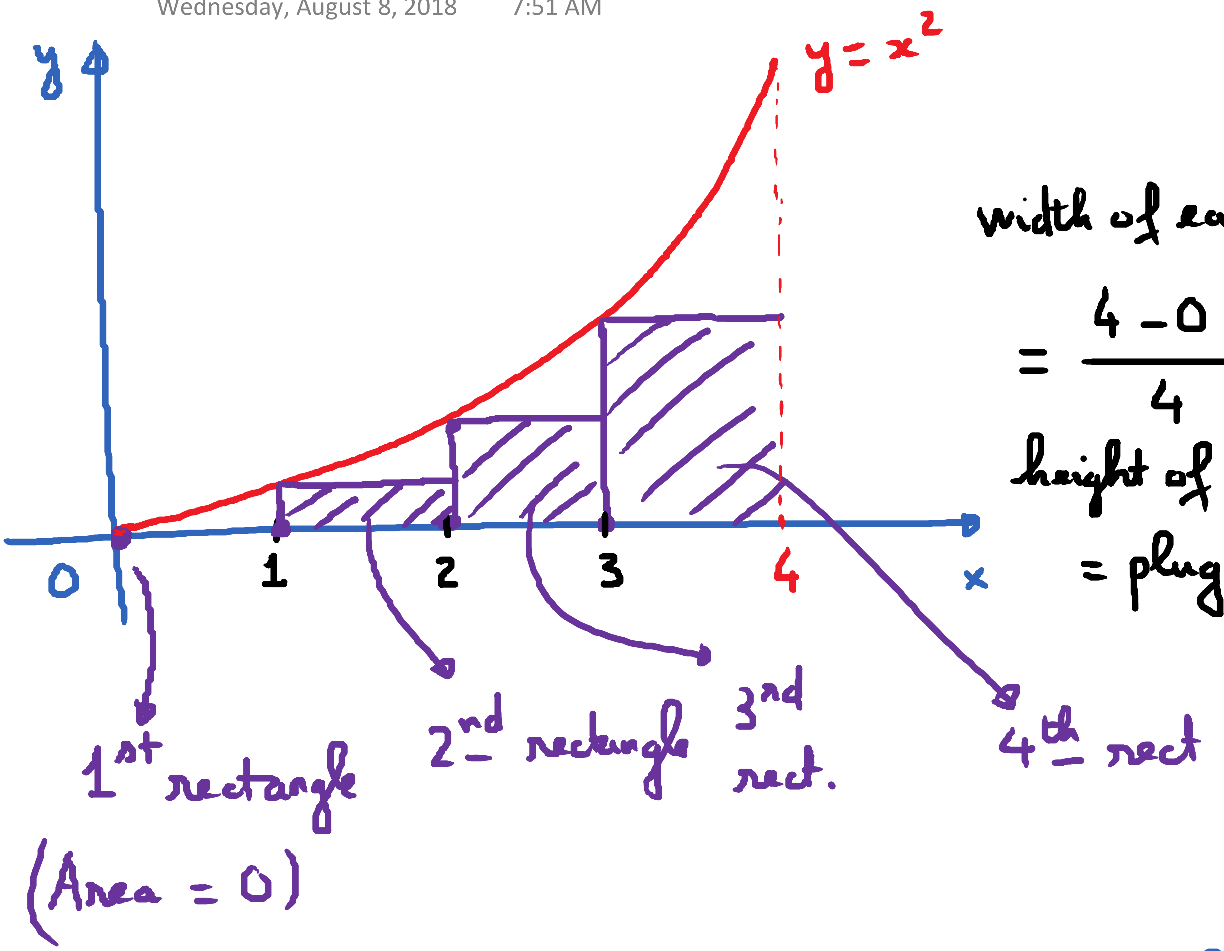
Find L_4



width of each rectangle

$$= \frac{4-0}{4} = 1$$

height of a rectangle

= plug a left endpoint to f

$$L_4 = \text{Sum of the areas of these 4 rectangles}$$

$$= \underbrace{f(0)\cdot 1}_{\text{area of }1^{st}} + \underbrace{f(1)\cdot 1}_{\text{area of }2^{nd}} + \underbrace{f(2)\cdot 1}_{\text{area of }3^{rd}} + \underbrace{f(3)\cdot 1}_{\text{area of }4^{th}}$$

$$= \big(f(0) + f(1) + f(2) + f(3)\big)\cdot 1$$

$$= (0 + 1 + 4 + 9)\cdot 1 = 14.$$

So, $L_4 = 14$

Exact area under the curve satisfies $A > L_4 = 14$

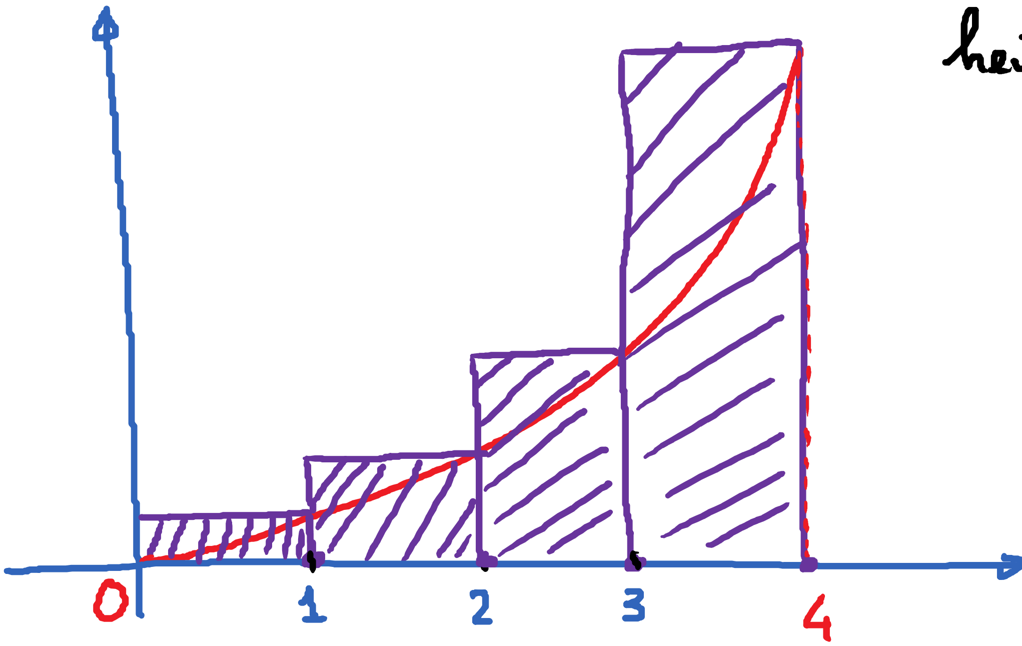
$14 < A$

underestimate for A

Find R_4

width = 1

height = plug a right endpoint into f



R_4 = Sum of the areas of these 4 rectangles

$$= \left(f(1) + f(2) + f(3) + f(4)\right) \cdot 1$$

$$= (1 + 4 + 9 + 16) \cdot 1 = 30$$

So, $R_4 = 30$.

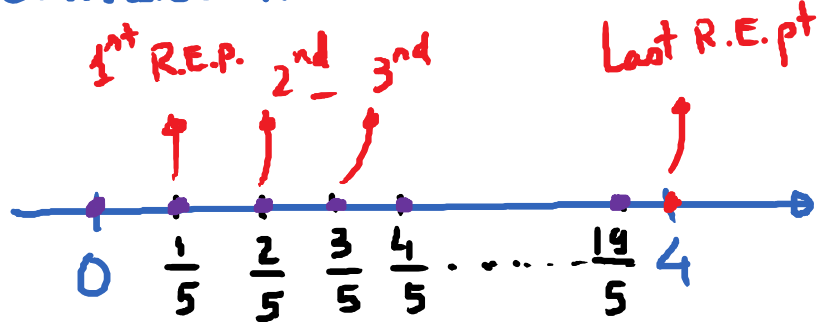
underestimate overestimate

We now know, $L_4 < A < R_4$

$$14 < A < 30$$

To get better approximations, we divide $[0,4]$ into more and more subintervals.

Consider $n = 20$ subintervals, what are L_{20} and R_{20}?



width of 1 subinterval:

$$\Delta x = \frac{4}{20} = \frac{1}{5}$$

$$L_{20} = \left(f(0) + f\left(\tfrac{1}{5}\right) + f\left(\tfrac{2}{5}\right) + \cdots + f\left(\tfrac{19}{5}\right)\right) \cdot \frac{1}{5}$$

$$R_{20} = \left(f\left(\tfrac{1}{5}\right) + f\left(\tfrac{2}{5}\right) + \cdots + f(4)\right) \cdot \frac{1}{5}$$

After calculation : $L_{20} = 19.76$, $R_{20} = 22.96$.

So, this tells us : $19.76 < A < 22.96$

* $n = 50$ subinterval. width of a subinterval $\Delta x = \frac{4}{50}$

$L_{50} = 20.6976$; $R_{50} = 21.9776$

→ Now, we know : $20.6976 < A < 21.9776$

→ First digit of the exact area A has to be 2

* $n = 100$ subinterval. width $\Delta x = \frac{4}{100}$.

$L_{100} = 21.0144$; $R_{100} = 21.6544$.

→ Now, we know: $21.0144 < A < 21.6544$

→ The first 2 digits of the exact area A has to be 21

→ $A \approx 21$ (something)

To find the exact area, take arbitrary n subintervals, write down the formulas for L_n, R_n (Left and Right Riemann sums with n subintervals), so

$$L_n < A < R_n$$

The take the limit $n \to \infty$. It turns out that:

$L = \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n$. So, By Squeeze Theorem,

$$A = L.$$

* Find R_n

width of a subinterval

$$\Delta x = \frac{4}{n}$$

1^{st} right end point : $\frac{4}{n}$

2^{nd} __________ : $\frac{8}{n}$

In general, the i^{th} right end point : $\frac{4i}{n}$ where i is a whole #.