4 : Last Right end pt

$$R_n = \underbrace{\left[f\left(\frac{4}{n}\right) + f\left(\frac{8}{n}\right) + f\left(\frac{12}{n}\right) + \cdots + f\left(\frac{4n}{n}\right)\right]}_{\text{Sum of the heights}} \cdot \frac{4}{n}$$

width

$$\underbrace{\sum_{i=1}^{10}}_{\text{Sigma or Summation notation}} i^2 = 1^2 + 2^2 + 3^2 + \cdots + 10^2$$

Using the Summation notation for the sum in R_n, we have:

$$R_n = \left[\sum_{i=1}^{n} f\left(\frac{4i}{n}\right)\right] \cdot \frac{4}{n}$$

(Recall that $f(x) = x^2$, so $f\left(\frac{4i}{n}\right) = \left(\frac{4i}{n}\right)^2 = \frac{16i^2}{n^2}$)

$$R_n = \frac{4}{n} \sum_{i=1}^{n} \frac{16i^2}{n^2} = \frac{64}{n^3} \boxed{\sum_{i=1}^{n} i^2}$$

Special summation

Formula: $\displaystyle\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

$\left(\text{E.g. } \displaystyle\sum_{i=1}^{100} i^2 = 1^2 + \cdots + 100^2 = \frac{100 \cdot (101)(201)}{6} \right)$

Apply the formula to our sum:

$$R_n = \frac{64}{n^{\cancel{3}2}} \cdot \frac{\cancel{n}(n+1)(2n+1)}{6}$$

$$R_n = \frac{64(n+1)(2n+1)}{6n^2}$$

$$\lim_{n \to \infty} R_n = \lim_{n \to \infty} \frac{64(n+1)(2n+1)}{6n^2} = \frac{128}{6} = \boxed{\frac{64}{3}}$$

Similarly, $\displaystyle\lim_{n \to \infty} L_n = \frac{64}{3}$

So, the exact A is $\boxed{\dfrac{64}{3}}$

Idea of Riemann Sums:

Use L_n, R_n to find the area under the curve $y = f(x)$ over $[a,b]$

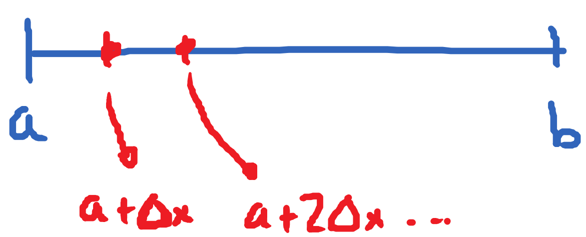
① Divide $[a,b]$ into n subintervals.

The width of each subinterval : $\Delta x = \dfrac{b-a}{n}$.

(This is the width of each small rectangle)

② Right Riemann Sum R_n :

$i^{\underline{th}}$ right endpoint : $x_i = a + i\Delta x$



$$R_n = \left[\sum_{i=1}^{n} f(x_i)\right]\cdot \Delta x = \frac{b-a}{n}\sum_{i=1}^{n} f\left(a + i\cdot\frac{b-a}{n}\right)$$

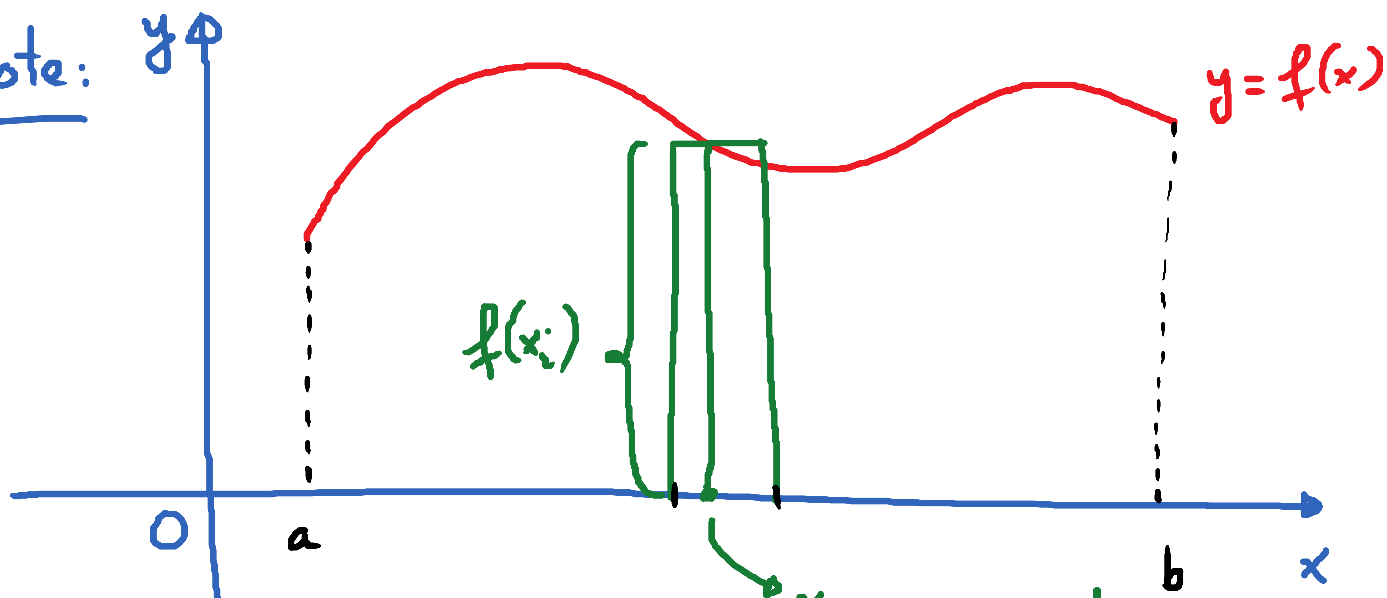
Left Riemann Sum

i^{th} left endpoint : $a + (i-1)\Delta x$

$$L_n = \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + (i-1)\cdot\frac{b-a}{n}\right)$$

③ Exact area $= \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n$.

Note:



x_i : any point in the i^{th} subinterval

$$\lim_{n\to\infty} \underbrace{\Delta x}_{\text{width of a rectangle}} \cdot \sum_{i=1}^{n} \underbrace{f(x_i)}_{\text{height of the } i^{\text{th}} \text{ rectangle}} = A = \text{exact area}$$

Very Important notation:

The limit above is called the definite integral of $f(x)$ on $[a,b]$ and it is denoted by

integrand

upper bound

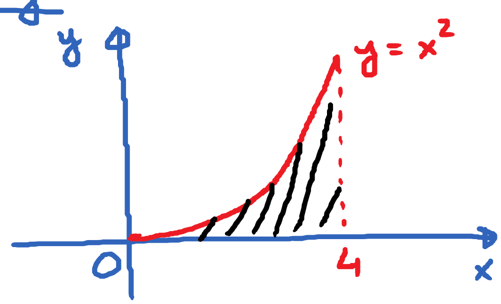
$$\int_a^b \boxed{f(x)}\,\boxed{dx} = \text{exact area under the curve } y = f(x) \text{ on } [a,b]$$

lower bound

variable of integration

This is read as: the definite integral from a to b of $f(x)$ with respect to x.

E.g. We have seen:



$$\int_0^4 x^2\,dx = \boxed{\frac{64}{3}}$$

area under $y = x^2$ on $[0,4]$

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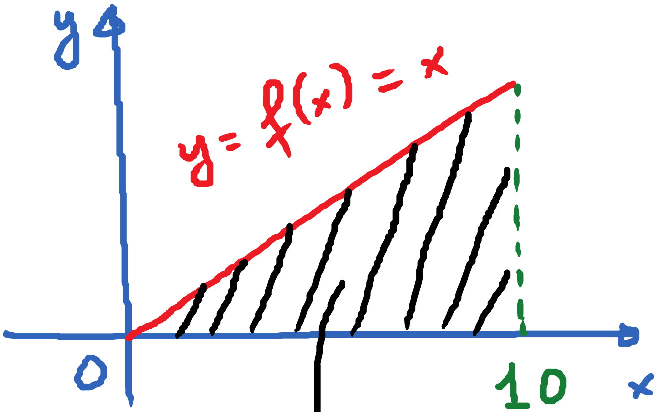
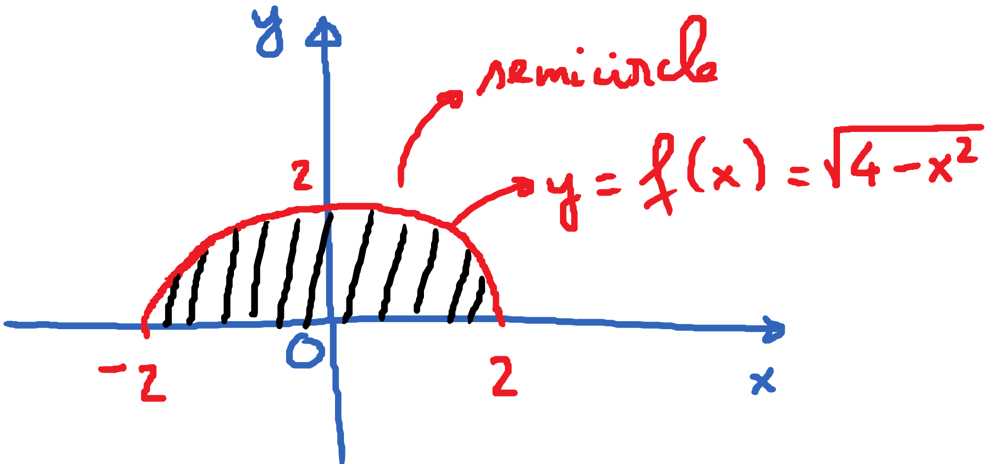


$$\int_0^{10} f(x)\,dx = 50$$

Area Triangle $= \frac{1}{2}\cdot 10\cdot 10$



$$\int_{-2}^{2}\sqrt{4-x^2}\,dx \overset{?}{=} \frac{\pi\cdot(2)^2}{2} = \boxed{2\pi}.$$

Area under the semicircle

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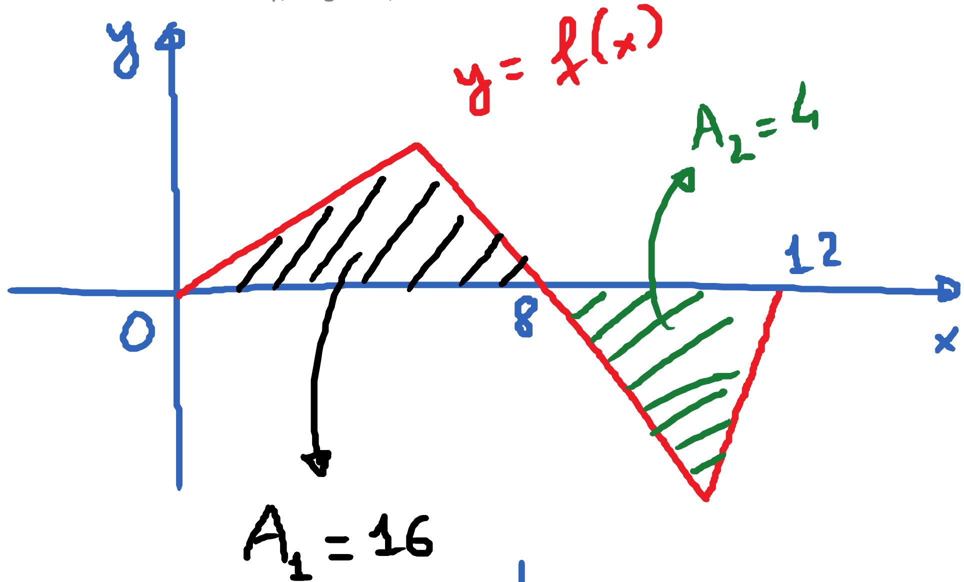
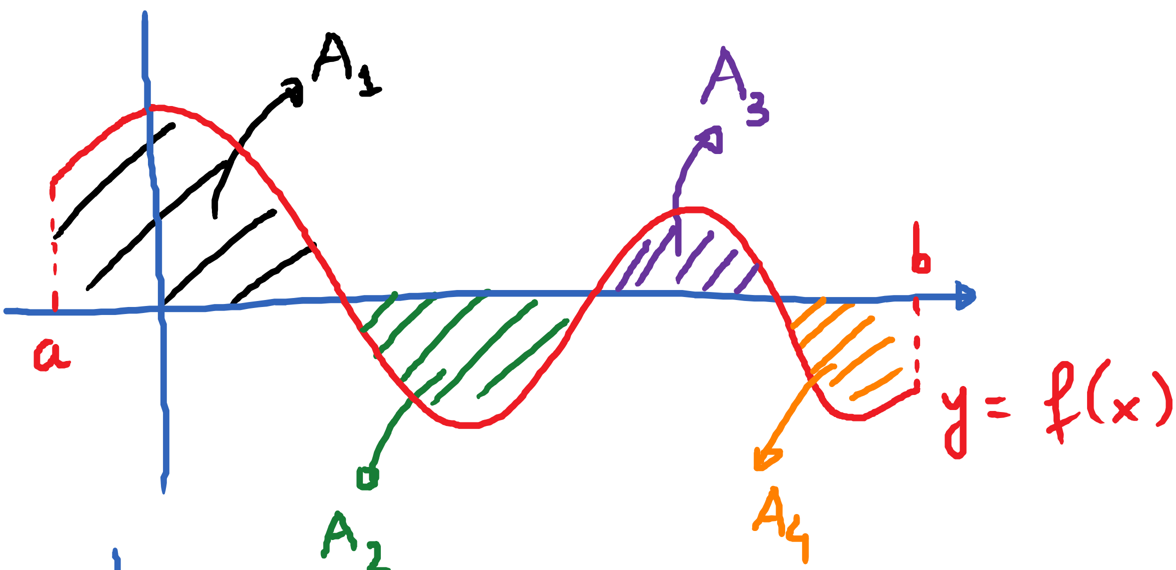


$$\int_0^{12} f(x)\,dx = A_1 - A_2$$
$$= 16 - 4 = 12$$

(We subtract A_2 because the function is below the x-axis)

So, in general, $\int_a^b f(x)\,dx$ = signed area between $f(x)$ and x-axis on $[a, b]$.



$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3 - A_4$$

Useful Properties of the definite integral:

1. $$\int_a^a f(x)\,dx = 0$$

2. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

E.g. $$\left(\int_1^3 f(x)\,dx = -\int_3^1 f(x)\,dx \right)$$

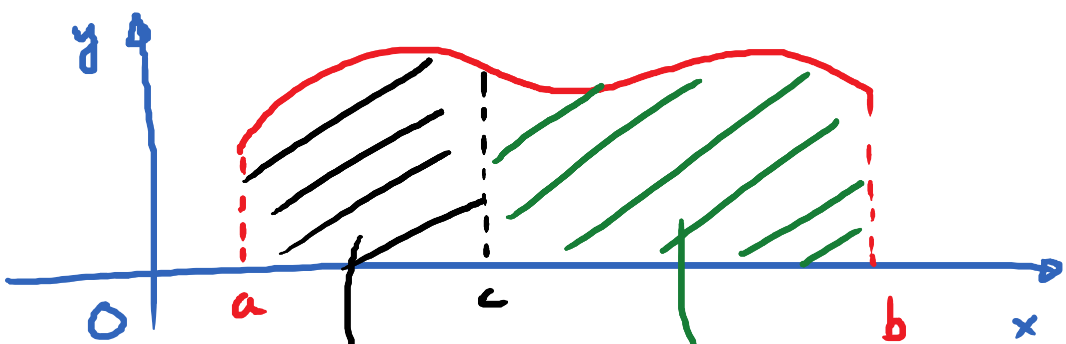
3. $$\int_a^b k \cdot f(x)\,dx = k \int_a^b f(x)\,dx$$

4. $$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x) \pm \int_a^b g(x)\,dx$$

linearity

5. $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\left(\underline{\text{E.g.}} \quad \int_1^{100} f(x)\,dx = \int_1^{70} f(x)\,dx + \int_{70}^{100} f(x)\,dx \right)$$



$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \text{total area}$$

$$= \int_a^b f(x)\,dx$$