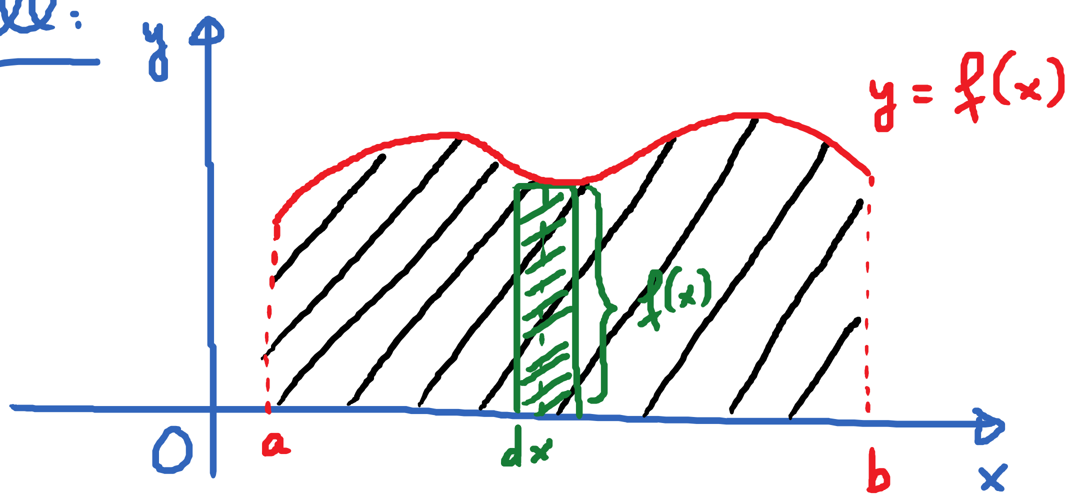
5.3. The Fundamental Theorem of Calculus

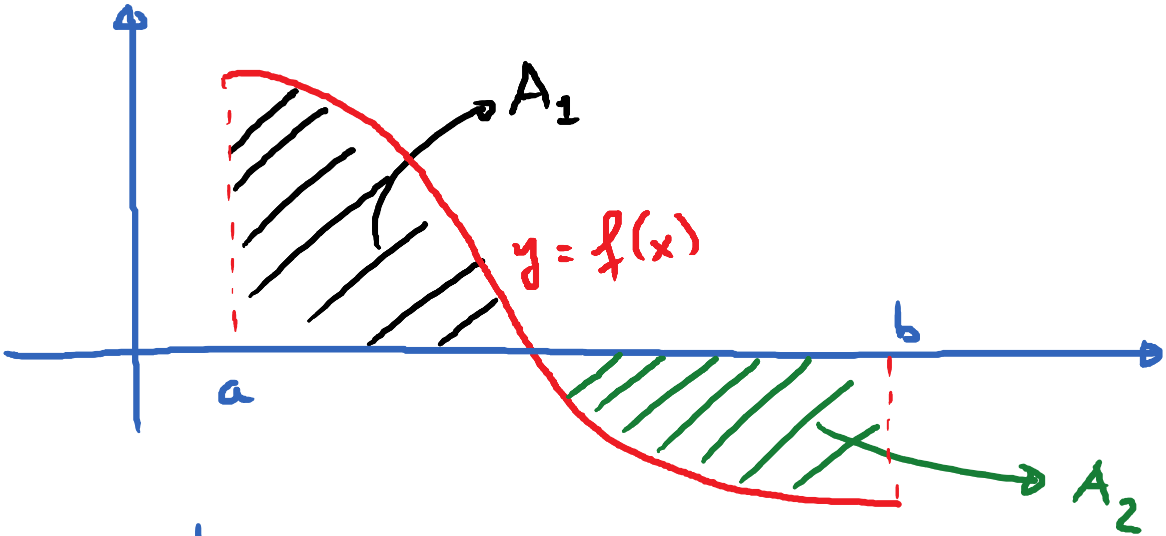
Wednesday, August 8, 2018 10:13 AM

Goals:
1. Apply the F.T.C. part II to find definite integrals
2. Apply the F.T.C part I to differentiate integrals.

Recall:



$$\int_a^b \underbrace{\boxed{f(x)}}_{\text{height}} \underbrace{\boxed{dx}}_{\text{width}} = \text{Area under the curve } y = f(x) \text{ over the interval } [a,b]$$



$$\int_a^b f(x)\,dx = A_1 - A_2 \quad \text{(signed area)}$$

→ F.T.C. Part II to find definite integrals.

E.g. $\displaystyle\int_3^7 x^2\,dx$

Step 1: Find the antiderivative.

$$\int x^2\,dx = \frac{x^3}{3} + C$$

(We used the formula : $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \,;\; n \neq -1$)

Since we are interested in the definite integral and not the antiderivative, we ignore the constant C and choose an antiderivative

$$F(x) = \frac{x^3}{3}.$$

Step 2: Evaluate $F(x)$ at the upper bound and lower bound of the definite integral and find the difference.

$$F(7) - F(3) = \frac{(7)^3}{3} - \frac{(3)^3}{3}$$

$$= \frac{343}{3} - \frac{27}{3} = \boxed{\frac{316}{3}}$$

So, $\boxed{\int_3^7 x^2\,dx = \frac{316}{3}}$

FTC - Part II

If f is a continuous function on $[a,b]$ and $F(x)$ is an antiderivative of $f(x)$; i.e, $F'(x) = f(x)$

then:

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

Very useful notation:

$$F(x)\Big|_a^b = F(b) - F(a)$$

$\longrightarrow$ FTC-II:

$$\boxed{\int_a^b f(x)\,dx = F(x)\Big|_a^b}$$

where F is an antiderivative of f.

E.g. $$\int_1^5 x^3\,dx = \left.\frac{x^4}{4}\right|_1^5 = \frac{(5)^4}{4} - \frac{(1)^4}{4}$$

$$= \frac{625}{4} - \frac{1}{4} = \frac{624}{4} = \boxed{156}$$

$$\int_6^{12} \sqrt{x}\,dx = \int_6^{12} x^{1/2}\,dx = \left.\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_6^{12}$$

Rewrite to have the form x^n

Anti power rule to find antiderivative

simplify, plug in

$$= \left.\frac{x^{3/2}}{3/2}\right|_6^{12} = \left.\frac{2}{3}x^{3/2}\right|_6^{12}$$

$$= \frac{2}{3}\left[(12)^{3/2} - (6)^{3/2}\right] = \ldots.$$

E.g.

$$\int_1^4 \frac{4}{x^2}\,dx = 4\cdot\int_1^4 \frac{1}{x^2}\,dx = 4\cdot\int_1^4 x^{-2}\,dx$$

Linearity

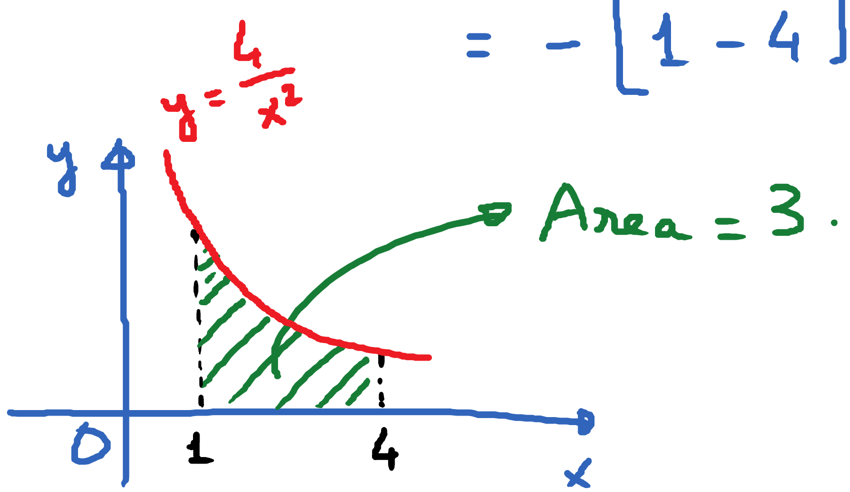
Rewrite

$$= 4\cdot\frac{x^{-2+1}}{-2+1}\Bigg|_1^4 = 4\cdot\frac{x^{-1}}{-1}\Bigg|_1^4$$

anti power rule

$$= -\frac{4}{x}\Bigg|_1^4 = -\left[\frac{4}{4}-\frac{4}{1}\right]$$

$$= -\left[1-4\right] = \boxed{3}.$$



Ex. Find the definite integrals

① $\displaystyle\int_4^8 \left(4t^{5/2} - 3t^{3/2}\right)dt$ ② $\displaystyle\int_0^{\pi/2} \sin\theta \, d\theta$

③ $\displaystyle\int_{\pi/4}^{\pi/2} \csc^2(\theta)\,d\theta$ * ④ $\displaystyle\int_1^4 \frac{2-\sqrt{u}}{u^2}\,du$

* ⑤ $\displaystyle\int_0^5 \sqrt{25-x^2}\,dx$ (Hint: Do not try to find the antiderivative here)

Sol: ① $\displaystyle = 4\cdot\int_4^8 t^{\frac{5}{2}}dt - 3\int_4^8 t^{\frac{3}{2}}dt$

$$= 4\cdot \frac{t^{\frac{7}{2}}}{\frac{7}{2}}\Bigg|_4^8 - 3\cdot\frac{t^{\frac{5}{2}}}{\frac{5}{2}}\Bigg|_4^8$$

$$= \left(\frac{8}{7}t^{\frac{7}{2}} - \frac{6}{5}t^{\frac{5}{2}}\right)\Bigg|_4^8 = \left[\frac{8}{7}(8)^{\frac{7}{2}} - \frac{6}{5}(8)^{\frac{5}{2}}\right]$$

$$- \left[\frac{8}{7}(4)^{\frac{7}{2}} - \frac{6}{5}(4)^{\frac{5}{2}}\right] = \ldots$$

(2) $$\int_0^{\pi/2} \sin\theta \, d\theta = -\cos(\theta)\Big|_0^{\frac{\pi}{2}} = -\left[\cos\left(\tfrac{\pi}{2}\right) - \cos(0)\right]$$

$$= -[0-1] = \boxed{1}$$

(3) $$\int_{\pi/4}^{\pi/2} \csc^2(\theta)\, d\theta = -\cot(\theta)\Big|_{\pi/4}^{\pi/2}$$

$$= -\left[\cot\left(\tfrac{\pi}{2}\right) - \cot\left(\tfrac{\pi}{4}\right)\right] = -[0-1] = \boxed{1}$$

(4) $$\int_1^4 \frac{2-\sqrt{u}}{u^2}\, du = \int_1^4 \left(\frac{2}{u^2} - \frac{\sqrt{u}}{u^2}\right) du$$

$$= \int_1^4 \left(2u^{-2} - \frac{u^{1/2}}{u^2}\right) du = \int_1^4 \left(2u^{-2} - u^{-3/2}\right) du$$

$$= \left(2 \cdot \frac{u^{-1}}{-1} - \frac{u^{-1/2}}{-1/2}\right)\Bigg|_1^4 = \left(-\frac{2}{u} + \frac{2}{\sqrt{u}}\right)\Bigg|_1^4$$

$$= \left(-\frac{2}{4} + \frac{2}{\sqrt{4}}\right) - \left(-\frac{2}{1} + \frac{2}{\sqrt{1}}\right) = -\frac{1}{2} + 1 + 2 - 2 = \boxed{\frac{1}{2}}.$$