5.5. Integration by Substitution or u-sub

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Recall:

$$\int x^2\,dx = \frac{x^3}{3} + C \;;\quad \int y^2\,dy = \frac{y^3}{3} + C$$

$$\int u^2\,du = \frac{u^3}{3} + C \;;\quad \int z^2\,dz = \frac{z^3}{3} + C$$

E.g. of u-sub

$$\int (\overset{u}{\boxed{7x+5}})^{10}\,dx$$

let $\boxed{u = 7x+5}$ $\rightarrow \dfrac{du}{dx} = 7 \rightarrow du = 7\,dx$

$\rightarrow \boxed{dx = \dfrac{du}{7}}$

$$\longrightarrow \int u^{10}\,\boxed{dx}$$

$$\longrightarrow \int u^{10}\cdot\frac{du}{7} = \frac{1}{7}\int u^{10}\,du = \frac{1}{7}\cdot\frac{u^{11}}{11} + C$$

$$= \frac{u^{11}}{77} + C.$$

Answer: $\boxed{\dfrac{(7x+5)^{11}}{77} + C}$

Strategy for u-sub.

(1) Select u. (2) Find $\frac{du}{dx}$ and solve for dx in terms of du

(3) Transform the original integral in terms of x to an integral in terms of the variable u only.

(4) Apply the basic antiderivative table

$$\int u^n \, du = \frac{u^{n+1}}{n+1} + C \; ; \; n \neq -1$$

$$\int \frac{1}{u} \, du = \ln|u| + C \, .$$

$$\int \sin(u)\,du = -\cos(u) + C$$

$$\int \cos(u)\,du = \sin(u) + C$$

$$\int e^u\,du = e^u + C$$

$$\int \frac{1}{1+u^2}\,du = \arctan(u) + C$$

etc.

E.g. Find $\displaystyle\int (\underbrace{\boxed{7x^2+5}}_{u})^{2018} \cdot x\, \boxed{dx}$

1. let $u = 7x^2+5$

2. $\dfrac{du}{dx} = 14x \rightarrow du = 14x\,dx \rightarrow dx = \boxed{\dfrac{du}{14x}}$

3. $\displaystyle\int u^{2018} \cdot \cancel{x} \cdot \frac{du}{14\cancel{x}} = \boxed{\int u^{2018} \frac{du}{14}} = \frac{1}{14}\int u^{2018}\,du$

integral in terms of u only

$$= \frac{1}{14} \cdot \frac{u^{2019}}{2019} + C = \frac{(7x^2+5)^{2019}}{28266} + C$$

E.g. $\int \boxed{x^2 \cdot \sin(x^3)}\, dx$

Let $u = x^3$. $du = 3x^2\,dx \longrightarrow dx = \dfrac{du}{3x^2}$

$$\longrightarrow \int \cancel{x^2} \cdot \sin(u) \cdot \frac{du}{3\cancel{x^2}} = \frac{1}{3}\int \sin(u)\, du$$

$$= -\frac{\cos(u)}{3} + C = \boxed{-\frac{\cos(x^3)}{3} + C}$$

Check: Take derivative of answer:

$$-\frac{-\sin(x^3)\cdot \cancel{3}x^2}{\cancel{3}} = x^2 \sin(x^3) \longrightarrow$$

same as integrand

$u = \ln x \; ; \; du = \frac{1}{x} dx$

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Ex. Find the antiderivative.

$\int u^2 du = \frac{u^3}{3} + C$

(1) $\int x \cdot e^{x^2+4} dx$

(2) $\int \frac{(\ln x)^2}{x} dx = \frac{(\ln x)^3}{3} + C$

$\sin(e^x) + C$

(3) $\int \boxed{e^x} \cos(\boxed{e^x}) \boxed{dx}$ u du

(4) $\boxed{\int \frac{\sin(t)}{\cos^3(t)} dt}$

(5) $\int y\sqrt{y^2 - 5}\, dy$ u

(6) $\int \sqrt{\cot(x)} \cdot \csc^2(x)\, dx$ u

(7) $\int \frac{\arctan(x)}{1 + x^2} dx$ u du

(1) $u = x^2 + 4 \; ; \quad du = 2x\,dx \; ; \quad dx = \frac{du}{2x}$

$$\int \cancel{x} \cdot e^u \cdot \frac{du}{2\cancel{x}} = \frac{1}{2} \int e^u du = \boxed{\frac{1}{2} e^{x^2+4} + C}$$

4 $\displaystyle\int \frac{\sin(t)}{\cos^3(t)}\,dt = \int \frac{\sin(t)}{\cos(t)} \cdot \frac{1}{\cos^2(t)}\,dt$

$\displaystyle = \int \underbrace{\boxed{\tan(t)}}_{u} \cdot \overbrace{\boxed{\sec^2(t)\,dt}}^{du}$

$u = \tan(t)$; $du = \sec^2(t)\,dt$

$\displaystyle = \int u\,du = \frac{u^2}{2} + C = \boxed{\frac{\tan^2(t)}{2} + C}$

2nd way: let $u = \cos(t)$; $du = -\sin(t)\,dt$

$\displaystyle \int \frac{-du}{u^3} = -\int \frac{du}{u^3} = -\int u^{-3}\,du = -\frac{u^{-2}}{-2} + C$

$\displaystyle = \frac{1}{2u^2} + C = \boxed{\frac{1}{2\cos^2(t)} + C}$

Ex. 7 $\displaystyle\int x \cdot (\underbrace{\boxed{2x+5}}_{u})^{2018}\ \boxed{dx} \rightarrow \frac{du}{2}$

8 $\displaystyle\int \frac{dx}{(1+\sqrt{x})^4}$

7 Let $u = 2x+5$. Then $du = 2dx$.

So, $dx = \dfrac{du}{2}$

$$\int x \cdot u^{2018} \cdot \frac{du}{2} = \frac{1}{2}\int \boxed{x} \cdot u^{2018}\, du$$

get x in terms of u

$u = 2x+5$. So, $x = \dfrac{u-5}{2}$

$$\frac{1}{2}\int \frac{u-5}{2} \cdot u^{2018}\, du = \frac{1}{4}\int (u-5)u^{2018}\, du$$

$$\frac{1}{4}\int \left(u^{2019} - 5u^{2018}\right) du = \frac{1}{4}\left(\frac{u^{2020}}{2020} - \frac{5u^{2019}}{2019}\right) + C$$

$$= \frac{u^{2020}}{8080} - \frac{5u^{2019}}{8076} + C$$