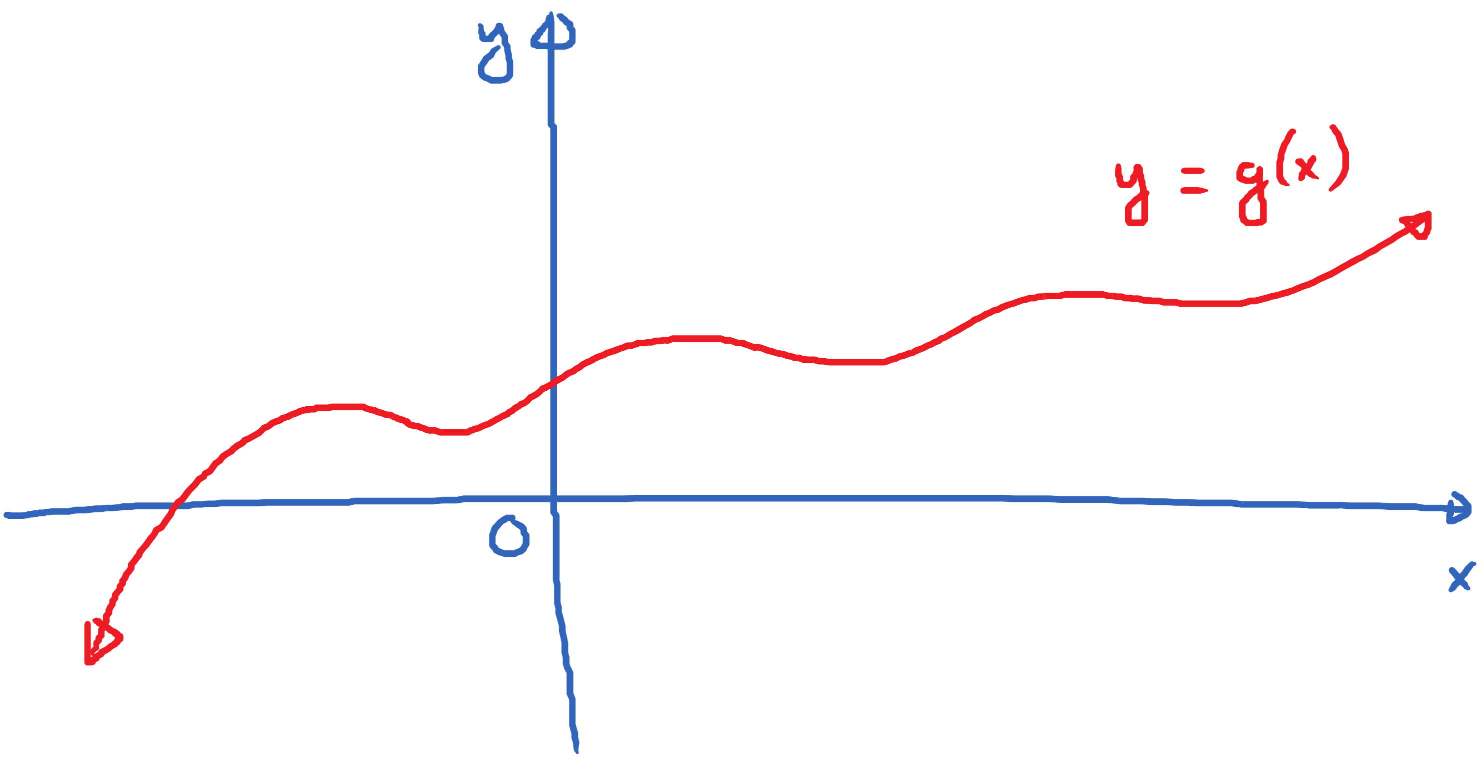
2.4 Continuity

Monday, July 16, 2018 9:43 AM

Goals: ① Understand the "limit" definition of continuity.

② Classify different types of discontinuity.

Intuitive concept of a continuous function.



What does it mean for $y = f(x)$ to be continuous at $x = a$?

To answer this, we analyze situations when f fails to be continuous at $x = a$.

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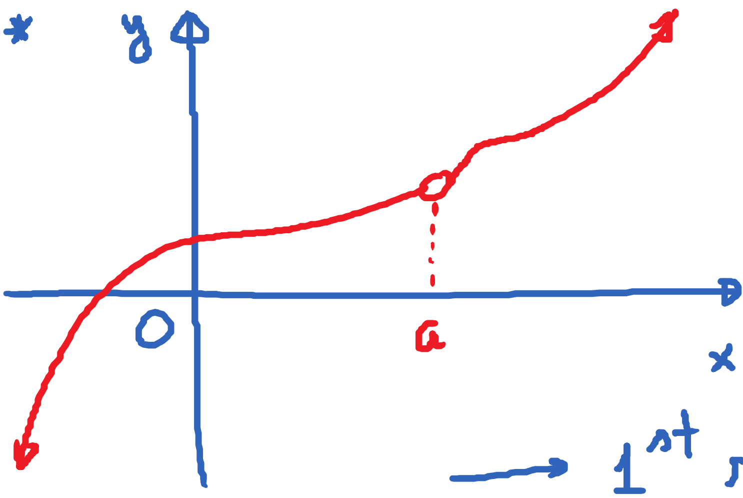

If $f(a)$ is undefined, then f is NOT continuous at $x = a$.

→ 1^{st} requirement for continuity:

$\boxed{f \text{ must be defined at } x = a}$

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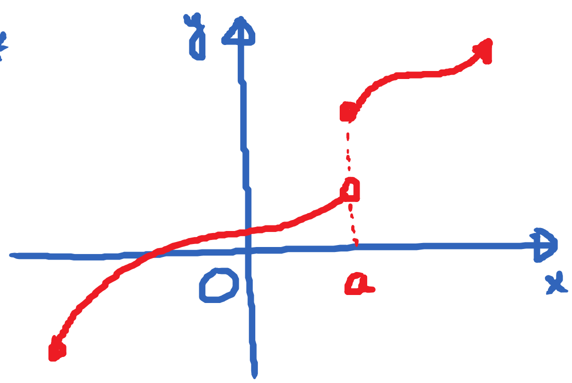

If $\lim_{x \to a} f(x)$ DNE, then f is NOT continuous at $x = a$.

→ 2nd requirement for continuity at $x = a$.

$$\boxed{\lim_{x \to a} f(x) \text{ must exist}}$$

(Note: $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$)

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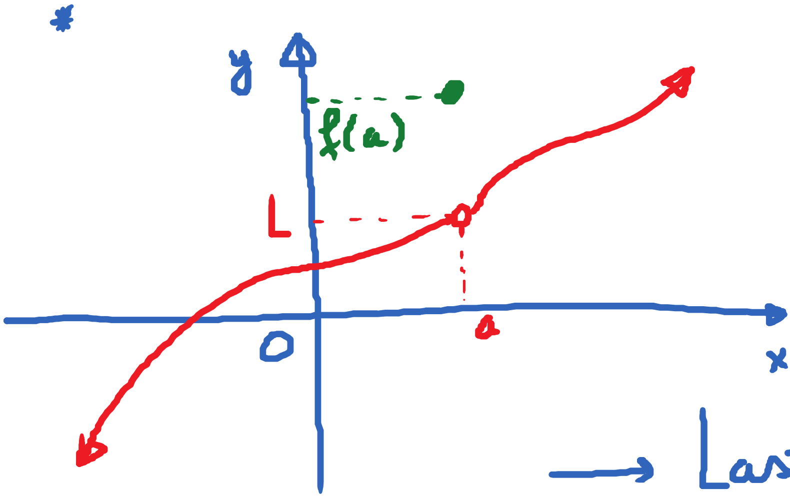


If $\lim_{x \to a} f(x) \neq f(a)$, then f is NOT continuous at $x = a$.

→ Last requirement for continuity at a point $x = a$.

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

Definition: A function $y = f(x)$ is continuous at a point $x = a$ if the following conditions are satisfied

1. $f(a)$ must be defined
2. $\lim_{x \to a} f(x)$ must exist
3. $\lim_{x \to a} f(x) = f(a)$

Note: We say that f is continuous on the open interval (c, d) if f is continuous at every point a within the interval

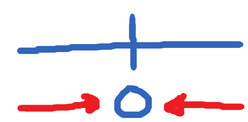
Note: If it is a half-closed interval, e.g., $[c, d)$, we look at the right limit at c.

E.g. let $f(x) = \begin{cases} x^2 - e^x & \text{if } x < 0 \\ x - 1 & \text{if } x \geq 0 \end{cases}$

From the definition of continuity above, is f continuous at $x = 0$?

✓ ① Is $f(0)$ defined? Yes, in fact, $f(0) = -1$

✓ ② Does $\lim_{x \to 0} f(x)$ exist?



$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} \left(x^2 - e^x\right) = (0)^2 - e^0$$
$$= -1$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} (x - 1) = -1$$

So, $\lim_{x \to 0} f(x)$ exists (b/c left limit = right limit)

✓ ③ $f(0) = \lim_{x \to 0} f(x)$? (both $= -1$)

Conclusion: f is continuous at $x=0$.

E.g. * Is $g(x) = \dfrac{2x^2-5x+3}{x-1}$ continuous at $x=1$? No. It violates (1)

* Is $u(x) = \begin{cases} 3x & \text{if } x<1 \\ x^3 & \text{if } x \geq 1 \end{cases}$ continuous at $x=1$? No. It violates (2)

* Is $w(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 3 & \text{if } x=0 \end{cases}$ continuous at $x=0$? No. It violates (3).

Fact: Any polynomial function is continuous at every real number, i.e., on $(-\infty,\infty)$

E.g. $g(x) = x^5 - 4x^4 + 3x^3 - 2x^2 + x - 1$.