3.2. The derivative as a function

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The formula $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

gives us the derivative of f at a particular point

$x = a$.

We want : to have a formula that can give us the derivative at arbitrary points.

$$x \longrightarrow \boxed{\;f'\;} \longrightarrow \text{derivative of } f \text{ at } x$$

First step : replace a by x in the definition :

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This defines a function, called f', in terms of x. That function $y = f'(x)$ is called the derivative function of f or in short it is called the derivative of f.

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{provided the limit exists}}$$

Domain of $f' = \{ x \mid f'(x) \text{ exists} \}$

E.g. $f(x) = x^2$

$$f'(x) = \lim_{h \to 0} \frac{\overbrace{f(\boxed{x+h})}^{(x+h)^2} - \overbrace{f(x)}^{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$\frac{x^2 - x^2}{h} = \frac{0}{0}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{- x^2}}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= \lim_{h \to \boxed{0}} (2x + h) = 2x$$

$f'(x) = 2x \rightarrow$ the derivative of the function $f(x) = x^2$ is given by the formula $\boxed{f'(x) = 2x}$

So, $f'(1) = 2$; $f'(2) = 4$; $f'\left(\frac{7}{2}\right) = 7$

$f'\left(\frac{9}{4}\right) = \frac{9}{2}$ . . . . .

Now, consider $f(x) = x^3$. We want to find the formula for $y = f'(x)$.

By definition,

$$f'(x) = \lim_{h \to 0} \frac{\overbrace{\boxed{f(\boxed{x+h})}}^{(x+h)^3} - \overbrace{\boxed{f(x)}}^{x^3}}{h}$$

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$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} \qquad \left(\frac{0}{0}\right)$$

Pascal Triangle:

1

$\boxed{1}$ $\boxed{1}$ $\qquad x + h$

1 2 1 $\qquad x^2 + 2xh + h^2$

$1 \to 3 \to 3 \to 1$ $\qquad \boxed{x^3 + 3x^2h + 3xh^2 + h^3}$

1 4 6 4 1

1 5 10 10 5 1

$$f'(x) = \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 \cancel{- x^3}}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}}$$

$$= \lim_{h \to 0} \left(3x^2 + 3xh + h^2\right) = 3x^2.$$

If $f(x) = x^3$, then $f'(x) = 3x^2$.

E.g. Equation of the tangent line to graph of $f(x) = x^3$ at the point where $x = -2$

Slope $\stackrel{?}{=} f'(-2) = 3 \cdot (-2)^2 = \boxed{12}$

Point $(-2, -8)$.

Equation: $y + 8 = 12(x + 2)$

$y = 12x + 16$

So far,

function $f(x)$	derivative $f'(x)$
$f(x) = x^2$	$f'(x) = 2x$
$f(x) = x^3$	$f'(x) = 3x^2$
$f(x) = x^4$	$f'(x) = 4x^3$
$f(x) = x^{2018}$	$f'(x) = 2018x^{2017}$

E.g. $f(x) = \sqrt{x}$

Use the definition of the derivative function to develop the formula for $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{\overset{\sqrt{x+h}}{\boxed{f(x+h)}} - \boxed{f(x)} \rightarrow \sqrt{x}}{h}$$

$$\left(\frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{x}}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

If $f(x) = \sqrt{x}$, then $f'(x) = \dfrac{1}{2\sqrt{x}}$.

Find $f'(100) = \dfrac{1}{20}$; $f'(81) = \dfrac{1}{18}$

$f'(0) =$ DNE ; $f'(-2)$ DNE.

<u>Def</u>: We say that a function f is differentiable at a point x if $f'(x)$ exists, i.e., the derivative exists at that point.