8.5. Rational Functions

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Objectives:

1. Rational Functions and their domains.
2. Asymptotes — Horizontal, Vertical
3. Graph Rational Functions & Applications

Rational Function

A rational function is a function f of the form

$$f(x) = \frac{p(x)}{q(x)}$$

where both $p(x)$ and $q(x)$ are polynomials.

E.g. $f(x) = \dfrac{2x+5}{2x-6}$; $g(x) = \dfrac{2x-11}{x^2+2x-8}$

$h(x) = \dfrac{5}{x-3}$; $u(x) = \dfrac{1}{x}$

To find the domain of a rational function $f(x) = \dfrac{p(x)}{q(x)}$.

Step 1: Set $q(x) = 0$ and solve for x.

Step 2: Domain = all real numbers except for the values in Step 1.

Conclusion: Express the answer in set builder notation or interval notation.

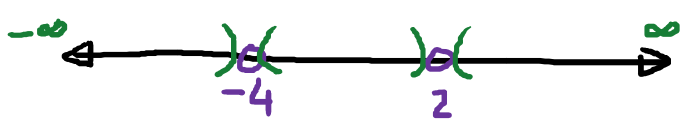
E.g. $f(x) = \dfrac{2x - 11}{x^2 + 2x - 8}$. Find domain of f.

Step 1: Set $x^2 + 2x - 8 = 0$

$$(x+4)(x-2) = 0 \rightarrow x = -4;\ x = 2$$

Step 2: Domain = all real #s except for -4 and 2

Conclusion: Interval Notation: $\boxed{(-\infty, -4) \cup (-4, 2) \cup (2, \infty)}$



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Set Builder Notation: $\boxed{\{x \mid x \neq -4 \text{ and } x \neq 2\}}$

Ex:

Function	Domain
$f(x) = \frac{1}{x}$	$(-\infty, 0) \cup (0, \infty)$ or $\{x \mid x \neq 0\}$
$f(x) = \frac{1}{x^2}$	$(-\infty, 0) \cup (0, \infty)$ or $\{x \mid x \neq 0\}$
$f(x) = \frac{x-3}{x^2+x-2} = \frac{x-3}{(x+2)(x-1)}$	$(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$ or $\{x \mid x \neq -2 \text{ and } x \neq 1\}$
$f(x) = \frac{2x+5}{2x-6} = \frac{2x+5}{2(x-3)}$	$(-\infty, 3) \cup (3, \infty)$ or $\{x \mid x \neq 3\}$

$$f(x) = \frac{x^2+2x-3}{x^2-x-2}$$

$$= \frac{x^2+2x-3}{(x+1)(x-2)}$$

$(-\infty,-1) \cup (-1,2) \cup (2,\infty)$

or

$\{x \mid x \neq -1, x \neq 2\}$

$$f(x) = \frac{-x^2}{x+1}$$

$(-\infty,-1) \cup (-1,\infty)$ or $\{x \mid x \neq -1\}$

$$f(x) = \frac{x}{x^2+4}$$

$x^2+4=0$

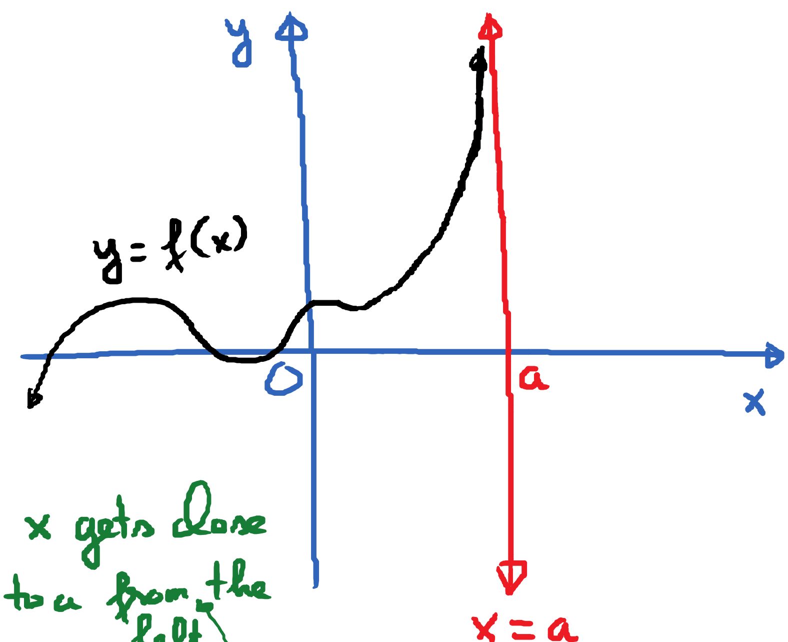
$x^2=-4$; $x = \pm 2i$

All real numbers

or $(-\infty,\infty)$

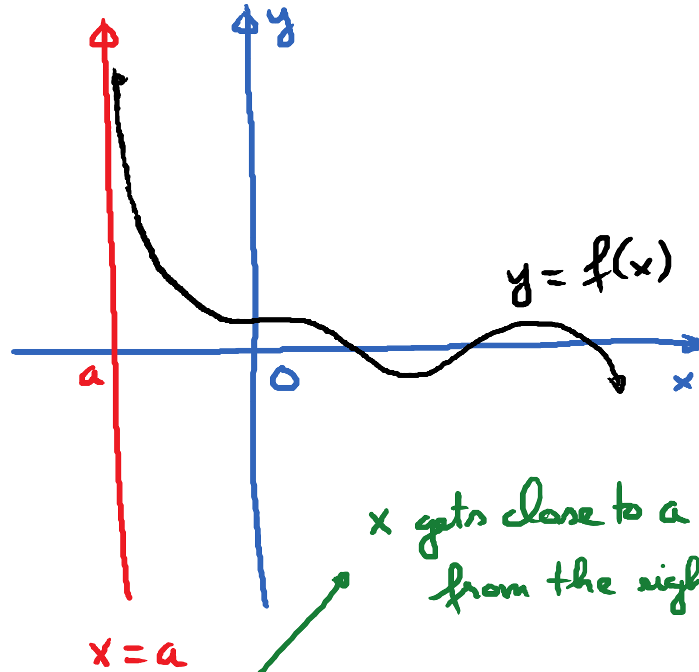
Asymptotes

Vertical Asymptotes:



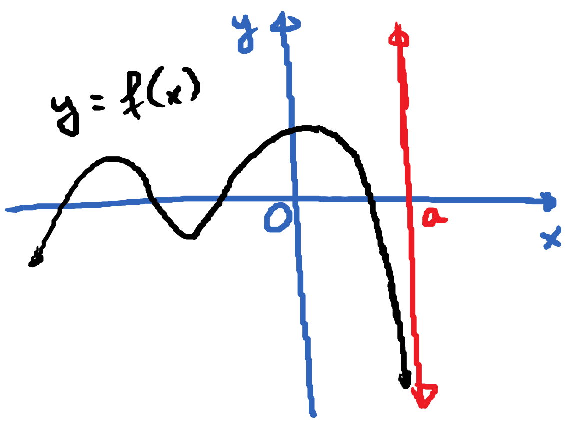
x gets close to a from the left

As $x \to a^{-}$, $f(x) \to \infty$

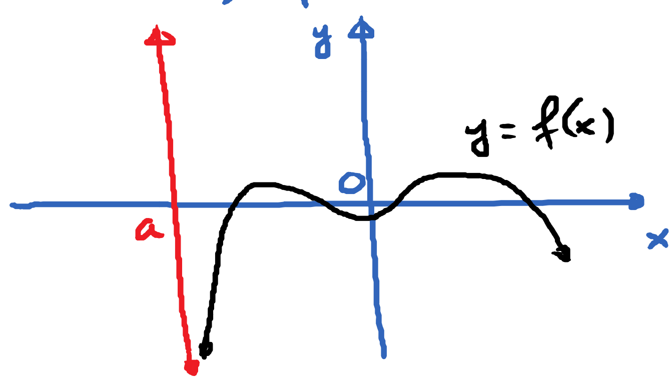


x gets close to a from the right

As $x \to a^{+}$, $f(x) \to \infty$



As $x \to a^{-}$, $f(x) \to -\infty$



As $x \to a^{+}$, $f(x) \to -\infty$

The line $x = a$ is a V.A. of the function $y = f(x)$ if one of above situations occur.

Q: How do we find the vertical asymptote(s) (V.A.) of a given rational function.

E.g. Find the V.A. (if any) of $f(x) = \dfrac{x^2 - 5x + 6}{x^2 - 9}$

Step 1: Factor.

Factor both top and bottom completely.

$$f(x) = \frac{(x-3)(x-2)}{(x-3)(x+3)}$$

Step 2: Cancel.

Cancel the common factor(s) of top and bottom.

$$f(x) = \frac{\cancel{(x-3)}(x-2)}{\cancel{(x-3)}(x+3)} = \boxed{\frac{x-2}{x+3}} \longrightarrow \text{Simplified Expression}$$

Step 3: Solve

Set the bottom of the simplified expression $= 0$ and solve for x.

$$x+3=0 \longrightarrow x=-3$$

Step 4: Vertical Asymptotes.

Result of step 3 gives us the V.A.

V.A. : $\boxed{x=-3}$

Step 5: Hole

The value(s) of x which make the canceled factor $=0$ are the x-value(s) of the hole(s) in the graph

$$\underbrace{\boxed{x-3}}_{\text{canceled factor}}=0 \longrightarrow \boxed{x=3} \longrightarrow \text{x-value of a hole in graph.}$$

Plug in Simplified expression

$$\boxed{\frac{x-2}{x+3}} \longrightarrow \frac{1}{6} \longrightarrow \overbrace{\boxed{\left(3, \frac{1}{6}\right)}}^{\text{Hole}}$$

E.g. Find V.A.(s) of $h(x) = \dfrac{x^2 - 4x}{x^3 - x}$

Step 1: Factor.

$$h(x) = \frac{x(x-4)}{x(x^2-1)} = \frac{x(x-4)}{x(x+1)(x-1)}$$

Step 2: Cancel

$$h(x) = \frac{\cancel{x}(x-4)}{\cancel{x}(x+1)(x-1)} = \boxed{\frac{x-4}{(x+1)(x-1)}}$$

Simplified Expression

Step 3: Solve

$(x+1)(x-1) = 0 \rightarrow x = -1$ and $x = 1$.

Step 4: V.A. $\boxed{x = -1}$ and $\boxed{x = 1}$

Step 5: Hole

$x = 0 \leftarrow$ x-coordinate of hole.

$$\frac{0-4}{(0+1)(0-1)} = \frac{-4}{1 \cdot (-1)} = 4.$$

Hole: $\boxed{(0, 4)}$