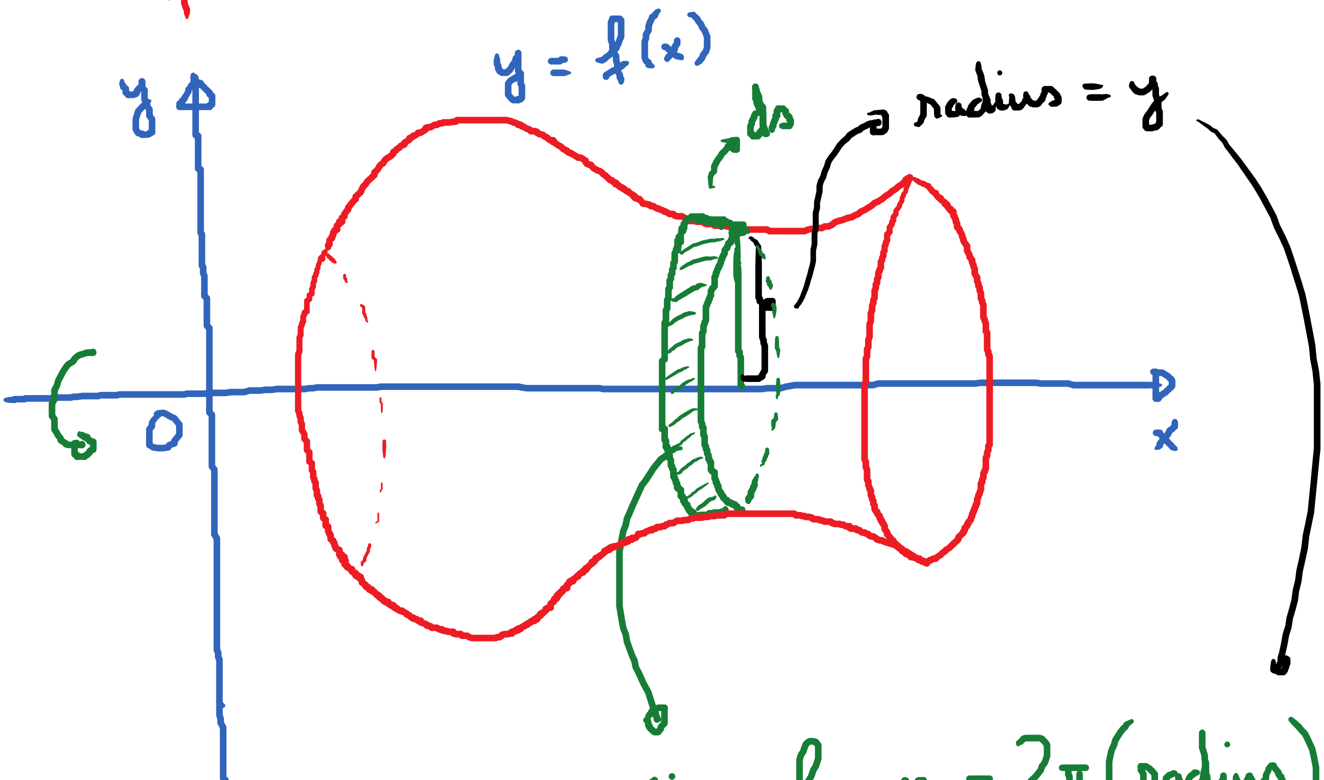
Thursday, September 6, 2018 8:51 AM

Surface Areas



circumference = 2π(radius)

$= 2\pi y$

Area of a piece of the surface $= 2\pi y \cdot ds$

$$\text{Surface Area} = \int_a^b 2\pi y\, ds = 2\pi \int_a^b y\, ds$$

There are 2 ways to make this integral computable

$$ds \begin{cases} ds = \sqrt{1+[f'(x)]^2}\,dx \\ ds = \underbrace{\sqrt{1+[g'(y)]^2}}\,dy \end{cases}$$

to obtain formula for g, we solve for x in terms of y in $y = f(x)$

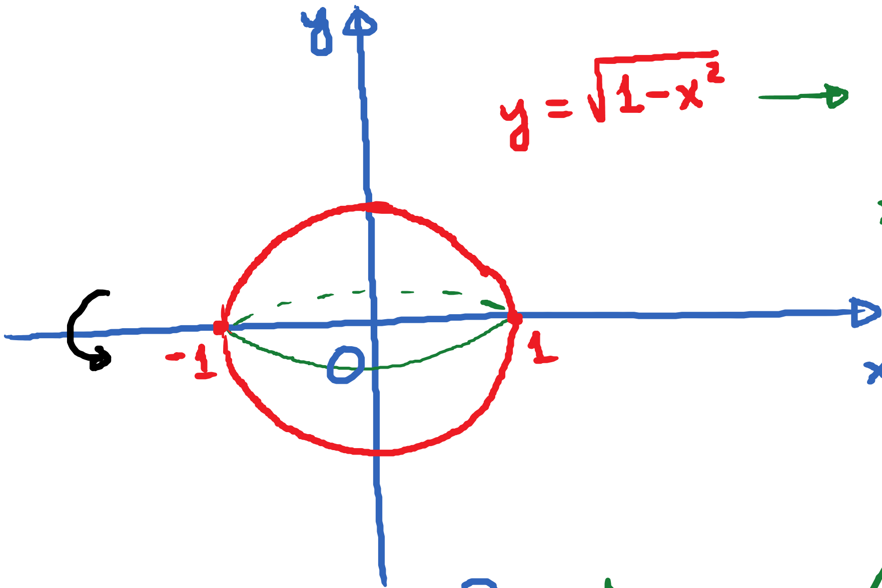
$$\text{I} \quad S = 2\pi \cdot \int_c^d y \cdot \sqrt{1 + g'(y)}\,dy$$ (c, d are the bounds for y)

(Note that in order to set up this integral you need to get x as a function of y)

$$\text{II} \quad S = 2\pi \cdot \int_a^b f(x) \cdot \sqrt{1+[f'(x)]^2}\,dx$$

(Here, we just need to replace y by $f(x)$)

E.g.



$y=\sqrt{1-x^2} \longrightarrow y^2 = 1-x^2$

$x^2 = 1-y^2$

Find surface area? Answer $= 4\pi$

$$S = 2\pi \int_{-1}^{1} y \, ds$$

II $\quad ds = \sqrt{1+[f'(x)]^2}\, dx$

$$y=\sqrt{1-x^2} \rightarrow f'(x) = \frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}} \cdot (-2x)$$

$$= (1-x^2)^{\frac{1}{2}}$$

$$= \frac{-2x}{2\sqrt{1-x^2}} = -\frac{x}{\sqrt{1-x^2}}$$

$$ds = \sqrt{1 + \left(-\frac{x}{\sqrt{1-x^2}}\right)^2}\,dx$$

$$= \sqrt{1 + \frac{x^2}{1-x^2}}\,dx = \sqrt{\frac{1 \cancel{-x^2} \cancel{+x^2}}{1-x^2}}\,dx$$

$$ds = \frac{1}{\sqrt{1-x^2}}\,dx$$

So, $$S = 2\pi \int_{-1}^{1} y\,ds = 2\pi \cdot \int_{-1}^{1} \cancel{\sqrt{1-x^2}} \cdot \frac{1}{\cancel{\sqrt{1-x^2}}}\,dx$$

$$= 2\pi \int_{-1}^{1} dx = 2\pi \cdot x \Big|_{-1}^{1} = \boxed{4\pi}.$$

Another way: do the integral w.r.t. y.

1st step: Solve for x in terms of y in the original formula.

$$y = \sqrt{1 - x^2} \longrightarrow y^2 = 1 - x^2$$

$$\longrightarrow x^2 = 1 - y^2 \longrightarrow x = \pm\sqrt{1 - y^2}$$

But by symmetry, we consider the part of the curve where $x = \sqrt{1-y^2}$ and then multiply the answer by 2 at the end

So, consider $x = \sqrt{1 - y^2}$

$$S = 2\pi \int_0^1 y \, ds$$

2nd step: find ds in terms of dy

$$ds = \sqrt{1 + [g'(y)]^2}\, dy$$

$$\underbrace{x}_{g(y)} = \sqrt{1-y^2} \longrightarrow g'(y) = \frac{-\cancel{2}y}{\cancel{2}\sqrt{1-y^2}} = -\frac{y}{\sqrt{1-y^2}}$$

$$ds = \sqrt{1 + \left(-\frac{y}{\sqrt{1-y^2}}\right)^2}\, dy$$

$$ds = \frac{1}{\sqrt{1-y^2}}\, dy$$

$$\frac{du}{-2y}$$

3rd step:

$$2\pi \int_0^1 y \cdot \frac{1}{\sqrt{1-y^2}}\, dy = 2\pi \cdot \int_0^1 \frac{y}{\sqrt{1-y^2}}\, dy$$

let $u = 1-y^2$; $du = -2y\, dy$. So, $dy = \dfrac{du}{-2y}$

$$= \cancel{2}\pi \cdot \int \frac{\cancel{y}}{\sqrt{u}} \cdot \frac{du}{-\cancel{2y}} = -\pi \int_1^0 \frac{du}{\sqrt{u}}$$

$$= \pi \cdot \int_0^1 u^{-\frac{1}{2}}\, du = \pi \cdot \frac{u^{\frac{1}{2}}}{\frac{1}{2}} \Bigg|_0^1 = 2\pi \cdot u^{\frac{1}{2}} \Bigg|_0^1$$

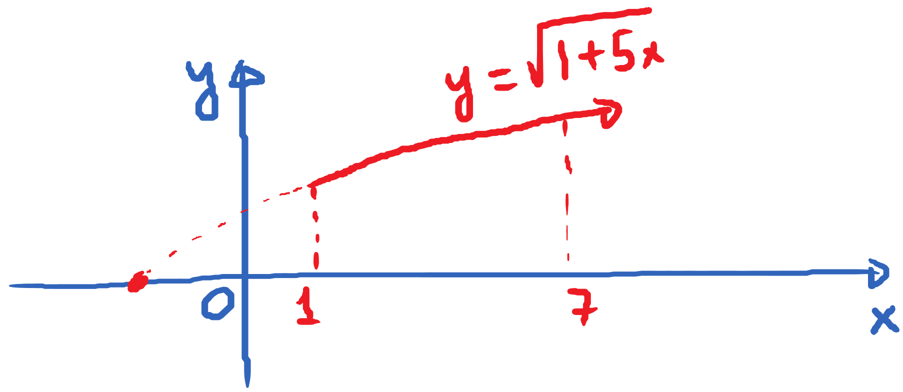
$$= 2\pi \cdot$$

$\longrightarrow$ Answer : $\boxed{4\pi}$.

Ex. Find area of the surface obtained by rotating

$$y = \sqrt{1+5x}, \quad 1 \le x \le 7$$

about the x-axis.



* Do this as an integral w.r.t. x

1st step: $ds = \sqrt{1 + [f'(x)]^2}\, dx$

$$f'(x) = \frac{dy}{dx} = \frac{5}{2\sqrt{1+5x}}$$