2.4. Arc Lengths and Surface Areas

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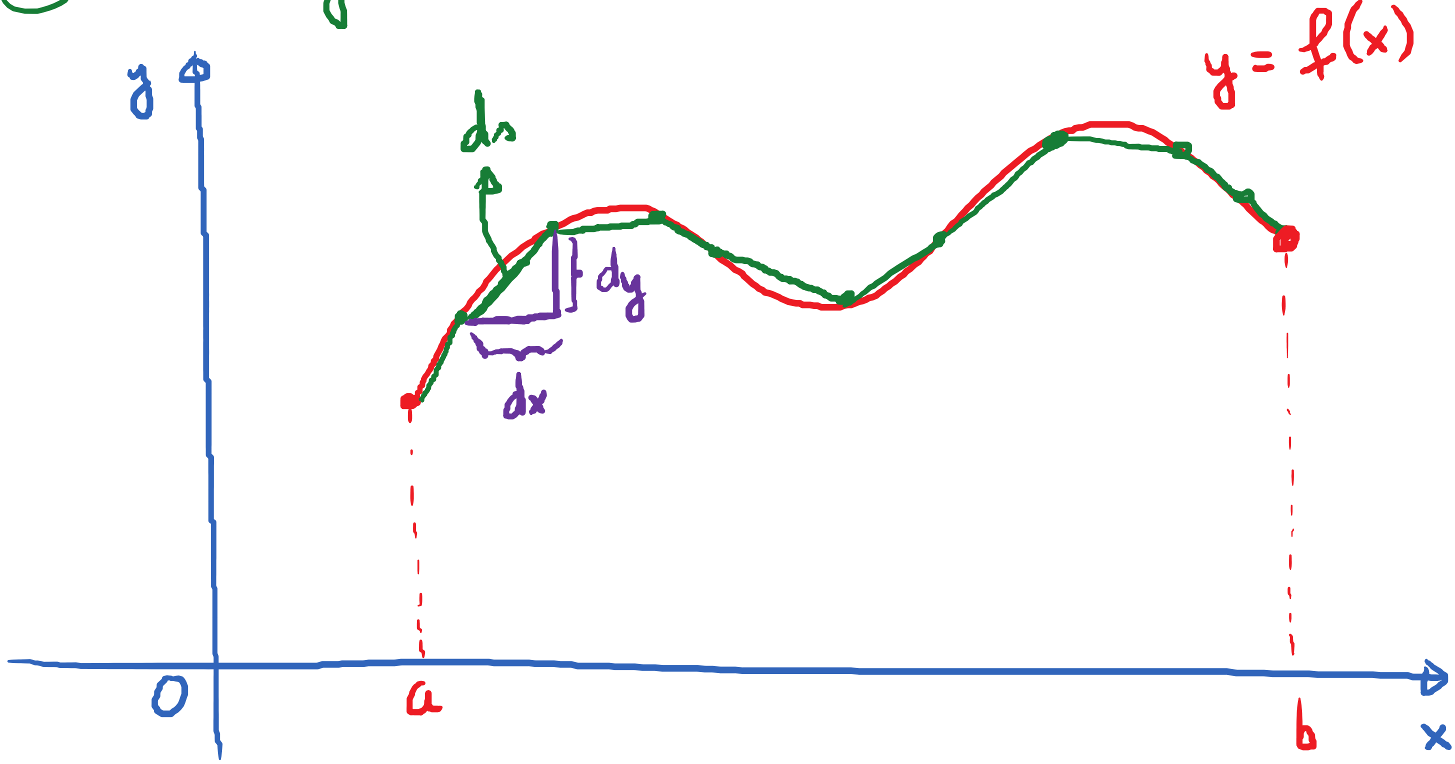
1 Arc length



Find the length of the curve $y = f(x)$, $a \leq x \leq b$

$$\boxed{L = \int_a^b ds = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx}$$

Where does this formula come from?

* Divide the curve into small pieces

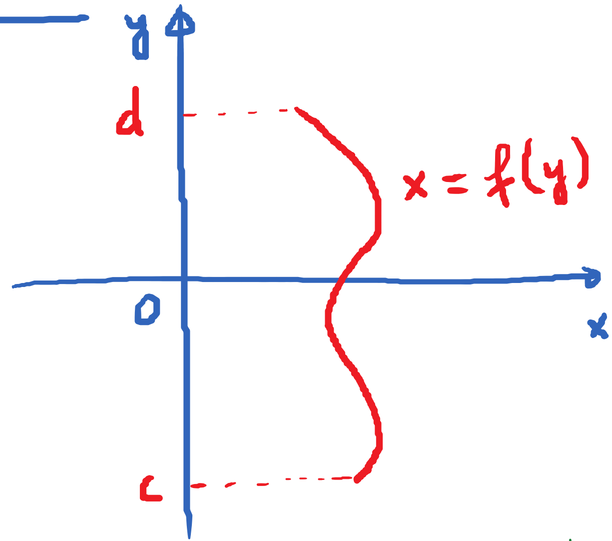
$$\underbrace{ds}_{\text{length of a small piece}} = \sqrt{(dx)^2 + (dy)^2} \quad \text{(By Pythagorean Theorem}$$

$$ds = \sqrt{(dx)^2 \left[1 + \left(\frac{dy}{dx}\right)^2\right]}$$

$$ds = \sqrt{1 + \Big(\underbrace{\boxed{\frac{dy}{dx}}}_{f'(x)}\Big)^2} \cdot dx$$

$$\rightarrow L = \underbrace{\boxed{\int_a^b}}_{\text{Sum}} \underbrace{ds}_{\text{small length}} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Note:



Length of curve from $x = c$ to $x = d$ is:

$$L = \int_c^d \sqrt{1 + [f'(y)]^2}\, dy$$

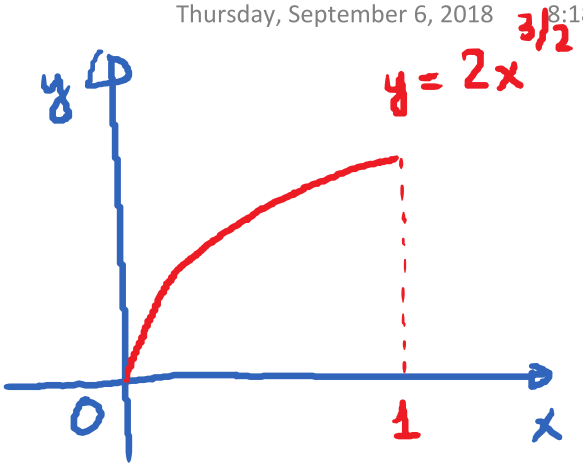
E.g.

If we use formula, then

$$L = \int_0^3 \sqrt{1 + 1}\; dx$$

$$= \int_0^3 \sqrt{2}\, dx = \sqrt{2} \int_0^3 dx$$

$$= \sqrt{2}\, x \Big|_0^3 = 3\sqrt{2}$$



$L = ?$

$$\left(\frac{d}{dx}(x^n) = nx^{n-1} \right)$$

$$y = 2x^{3/2}$$

$$\frac{dy}{dx} = 2 \cdot \frac{3}{2} \cdot x^{\frac{3}{2}-1} = 3x^{\frac{1}{2}} = 3\sqrt{x}$$

$$L = \int_0^1 \sqrt{1 + (3\sqrt{x})^2}\, dx = \int_{\boxed{0}}^{\boxed{1}} \sqrt{\underbrace{1 + 9x}_{u}}\, dx$$

$\frac{du}{9}$

Let $u = 1 + 9x$. Then $du = 9\,dx$. So, $dx = \frac{du}{9}$

New bounds: $u = 1$ is the lower bound
$u = 10$ is the upper bound

$$\longrightarrow \int_1^{10} \sqrt{u}\, \frac{du}{9} = \frac{1}{9} \int_1^{10} \sqrt{u}\, du = \frac{1}{9} \int_1^{10} u^{\frac{1}{2}}\, du$$

$$= \frac{1}{9} \cdot \frac{2}{3} \cdot u^{\frac{3}{2}} \Big|_1^{10} = \frac{2}{27} \cdot \left((10)^{\frac{3}{2}} - 10 \right)$$

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Ex. Find the length of the curve:

$$y = \ln(\sec(x)) \;;\; 0 \le x \le \frac{\pi}{3}.$$

Step 1: $\dfrac{dy}{dx} = \dfrac{\cancel{\sec(x)} \cdot \tan(x)}{\cancel{\sec(x)}} = \tan(x)$

$$\left(\frac{d}{dx}\left(\ln(u)\right) = \frac{u'}{u} \right)$$

Step 2: $L = \displaystyle\int_0^{\pi/3} \sqrt{1 + \tan^2(x)}\, dx$

$\sec^2(x)$

Recall: $1 + \tan^2(x) = \sec^2(x)$

Why? $\boxed{\sin^2(x) + \cos^2(x) = 1}$

Divide both sides by $\cos^2(x)$

$$\tan^2(x) + 1 = \sec^2(x)$$

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$$L = \int_0^{\pi/3} \sqrt{\sec^2(x)}\, dx = \int_0^{\pi/3} \sec(x)\, dx$$

Formula: $\int \sec(x)\, dx = \ln|\sec(x) + \tan(x)| + C$

Where does this come from?

$$\int \frac{\sec(x)}{1} \cdot \frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)}\, dx$$

$$= \int \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x) + \tan(x)}\, dx$$

→ du

u

$$= \int \frac{du}{u} = \ln|u| + C = \ln|\sec(x) + \tan(x)| + C$$

$$L = \ln\left|\sec(x) + \tan(x)\right| \Big|_0^{\frac{\pi}{3}}$$

$$= \ln\left|\sec\left(\tfrac{\pi}{3}\right) + \tan\left(\tfrac{\pi}{3}\right)\right| - \ln\left|\sec(0) + \tan(0)\right|$$

$$= \boxed{\ln\left(2+\sqrt{3}\right)} - \cancel{\ln(1)} \to 0$$

Ex. Find the length of the curve

$$y = \int_1^x \sqrt{t^3 - 1}\, dt \quad ; \quad 4 \le x \le 9.$$

<u>Step 1:</u> $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\int_1^x \sqrt{t^3-1}\, dt\right)$

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$$= \sqrt{x^3 - 1}$$

Step 2:

$$L = \int_4^9 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_4^9 \sqrt{1 + \left(\sqrt{x^3 - 1}\right)^2}\, dx$$

$$= \int_4^9 \sqrt{1 + x^3 - 1}\, dx = \int_4^9 \sqrt{x^3}\, dx$$

$$= \int_4^9 x^{\frac{3}{2}}\, dx = \frac{2x^{\frac{5}{2}}}{5}\Bigg|_4^9 = \frac{2}{5}\left((9)^{\frac{5}{2}} - (4)^{\frac{5}{2}}\right)$$

$$= \frac{2}{5}\left((3)^5 - (2)^5\right) = \frac{2}{5} \cdot 211 = \boxed{\frac{422}{5}}$$