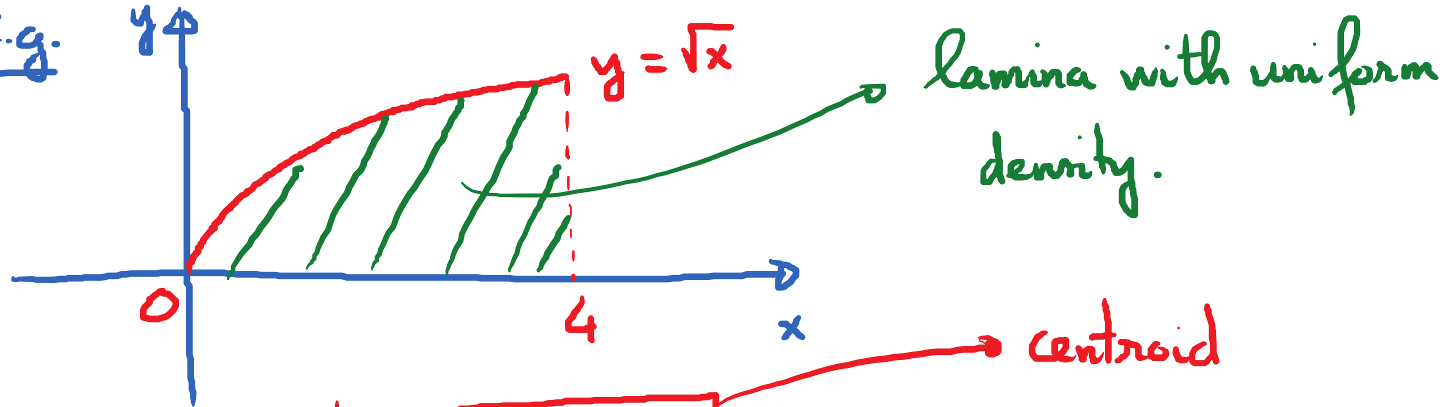
E.g.



Q: Find the $\boxed{\text{center of mass}}$ $(\bar{x}, \bar{y})$

Answer: centroid $= \left(\frac{12}{5}, \frac{3}{4}\right)$

$$m = \int_0^4 \sqrt{x}\,dx = \int_0^4 x^{\frac{1}{2}}\,dx = \frac{2x^{\frac{3}{2}}}{3}\Big|_0^4 = \frac{2(4)^{\frac{3}{2}}}{3} = \frac{16}{3}$$

$$M_y = \int_0^4 x\cdot\sqrt{x}\,dx = \int_0^4 x^{\frac{3}{2}}\,dx = \frac{2x^{\frac{5}{2}}}{5}\Big|_0^4 = \frac{2(4)^{\frac{5}{2}}}{5} = \frac{64}{5}$$

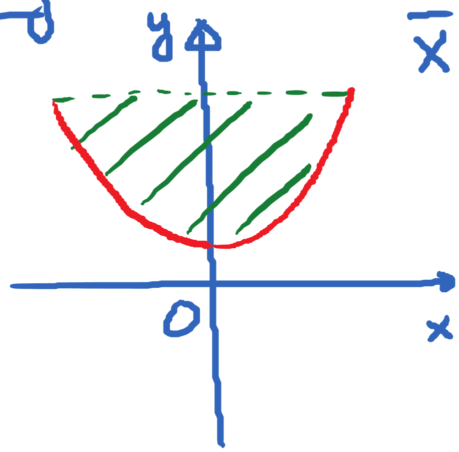
$$M_x = \frac{1}{2}\int_0^4 (\sqrt{x})^2\,dx = \frac{1}{2}\int_0^4 x\,dx = \frac{x^2}{4}\Big|_0^4 = 4$$

$$\bar{x} = \frac{M_y}{m} = \frac{64/5}{16/3} = \boxed{\frac{12}{5}}\ ;\quad \bar{y} = \frac{M_x}{m} = \frac{4}{16/3} = \boxed{\frac{3}{4}}$$

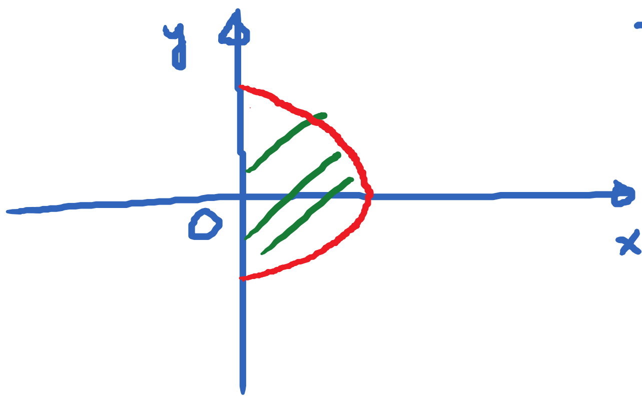
The symmetry principle:

If a lamina $\mathcal{R}$ is symmetric with respect to a line L, then the centroid must lie on L.

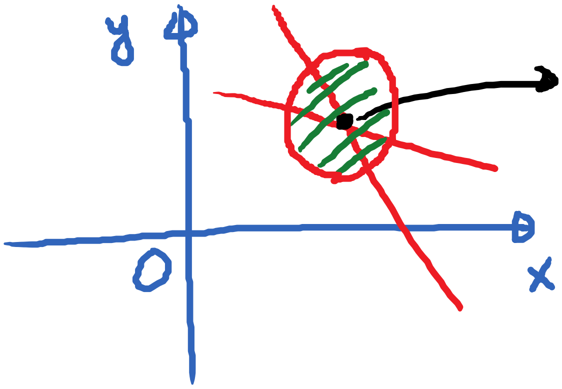
E.g.



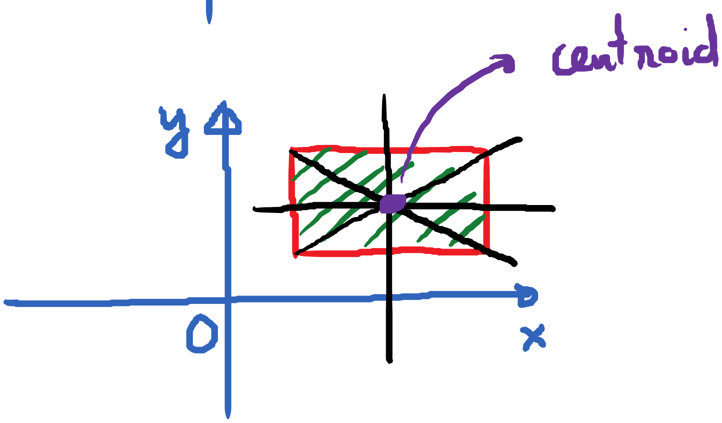
$\bar{x} = 0$ w/o calculation

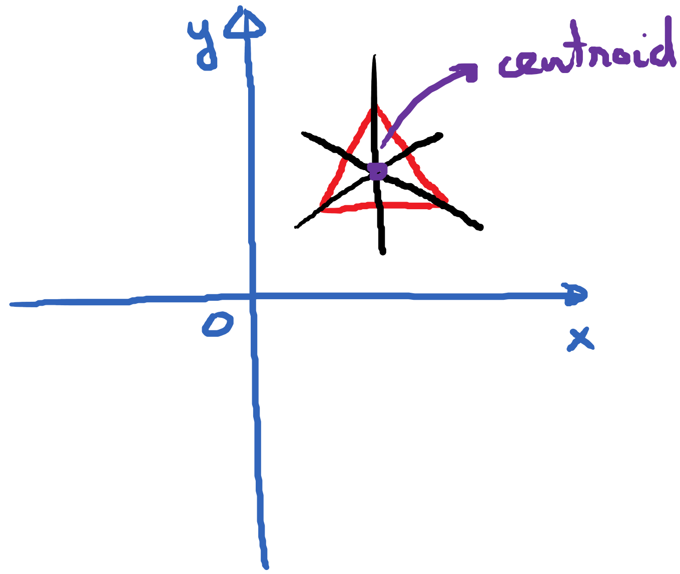


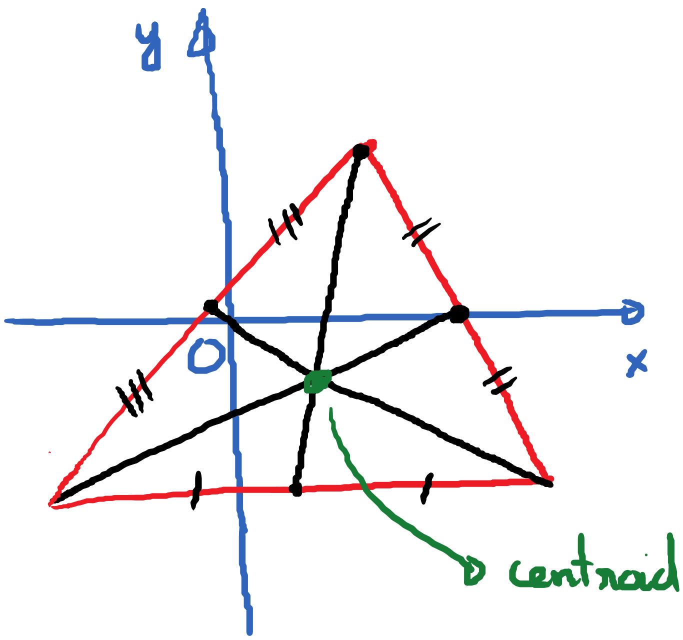
$\bar{y} = 0$ w/o calculation.



centroid = center of circle.



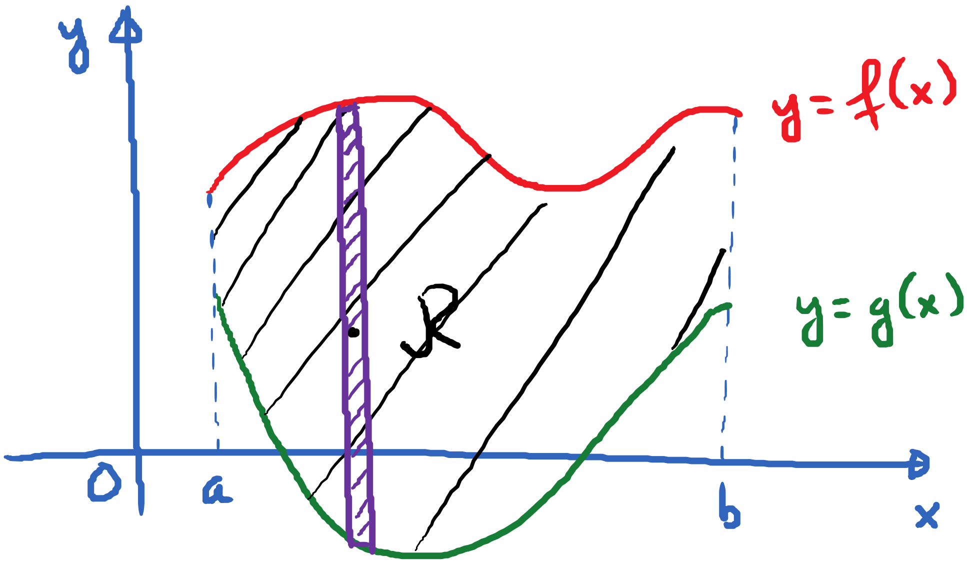




Thursday, September 13, 2018 9:46 AM

* Centroid of lamina bounded by 2 curves

(Assume uniform density)



Find the formula for centroid $(\overline{x}, \overline{y})$

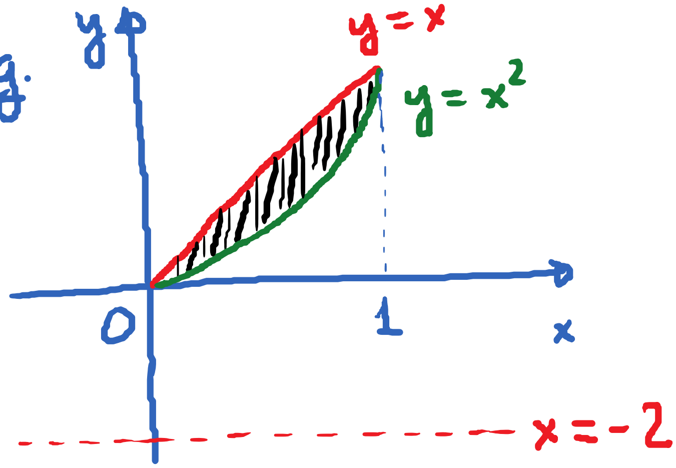
$$\overline{x} = \frac{M_y}{m} \; ; \quad \overline{y} = \frac{M_x}{m}$$

$$m = \rho \cdot \int_a^b \left(f(x) - g(x) \right) dx$$

$$M_y = \rho \int_a^b x \left(f(x) - g(x) \right) dx$$

$$M_x = \rho \int_a^b \frac{1}{2} \left[(f(x))^2 - (g(x))^2 \right] dx$$

E.g.



Find the centroid of this lamina.

$$m = \int_0^1 (x - x^2)\,dx$$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \boxed{\frac{1}{6}}$$

$$M_y = \frac{1}{12} \; ; \quad M_x = \frac{1}{15}$$

$$\bar{x} = \frac{M_y}{m} = \frac{1}{2} \; ; \quad \bar{y} = \frac{M_x}{m} = \frac{2}{5}$$

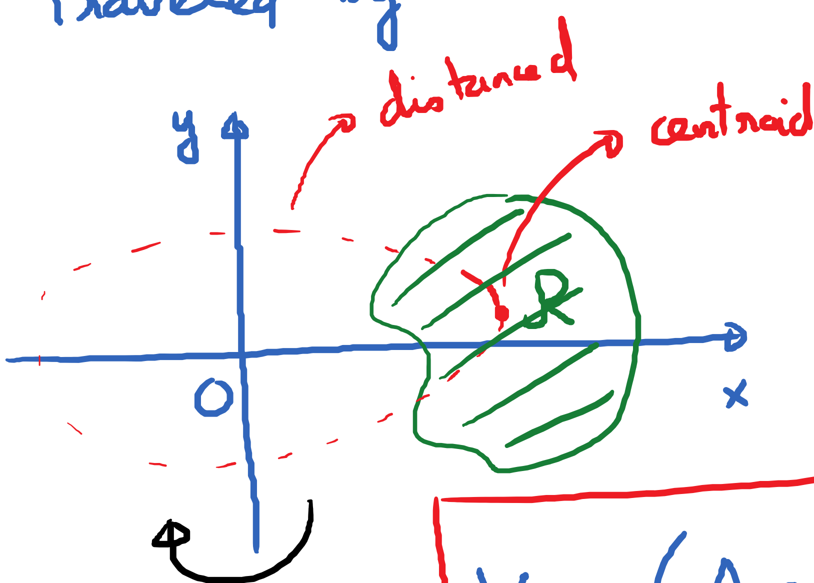
Answer: centroid $\left(\frac{1}{2}, \frac{2}{5}\right)$

$$V_y = \underbrace{\frac{1}{6}}_{\text{area}} \cdot \underbrace{2\pi \cdot \frac{1}{2}}_{d} = \frac{\pi}{6} \; ; \quad V_x = \frac{1}{6} \cdot 2\pi \cdot \frac{2}{5} = \frac{2\pi}{15}$$

$$V_{x=-2} = \frac{1}{6} \cdot 2\pi \cdot \left(2 + \frac{2}{5}\right) = \frac{4\pi}{5}.$$

* Applications of Centroid in finding volume of solid of revolution - Theorem of Pappus.

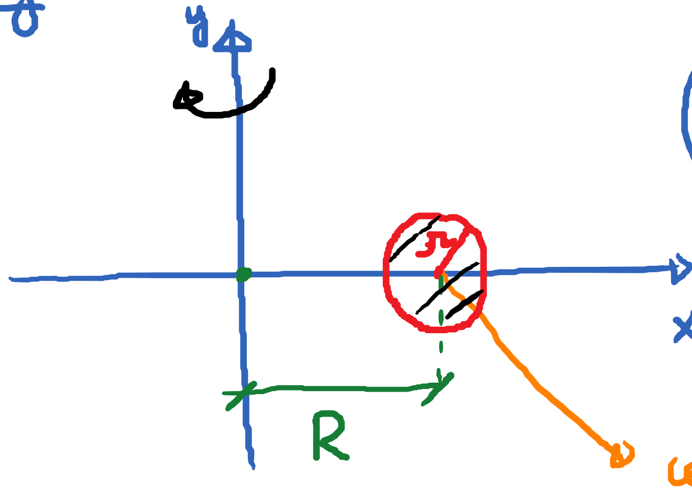
If a solid S is obtained by rotating a region R about an axis, then the volume of S is equal to the product of the area of R and the distance traveled by the centroid of R.

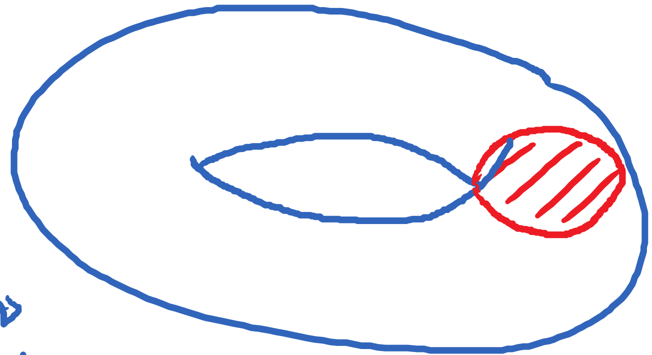


distance traveled by the centroid of R

$$\boxed{V_S = (\text{Area of } R)\cdot d}$$

E.g. Volume of solid torus





centroid: $(R, 0)$

$V_S = ?$

Area of region $= \pi \cdot (r)^2$

Distance traveled by centroid $= 2\pi R$

$$V_S = \pi \cdot (r)^2 \cdot 2\pi R = \boxed{2\pi^2 r^2 R}$$