3.1 Integration by Parts

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Product Rule (Cal 1)

$$(fg)' = fg' + f'g$$

Take the antiderivative of both sides:

$$fg = \boxed{\int fg'\,dx} + \boxed{\int f'g\,dx}$$

$$\boxed{\int \underbrace{f}_{u}\,\overbrace{\boxed{g'\,dx}}^{dv} = \underbrace{f}_{u}\underbrace{g}_{v} - \int \underbrace{g}_{v}\,\underbrace{\boxed{f'\,dx}}_{du}}$$

Let $f = u$; $g = v$

Then : $\underbrace{g'dx}_{\text{differential}} = dv$; $f'dx = du$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

E.g. Find $\int \underbrace{x}_{u} \underbrace{\sin x \, dx}_{dv}$

Integration by parts

Let $\begin{cases} \boxed{u = x} \\ dv = \sin x \, dx \end{cases}$ Need $\begin{cases} du = ? \\ v = ? \end{cases}$

$\begin{cases} du = dx \\ v = \int \sin x \, dx = \boxed{-\cos x} \end{cases}$

→ Plug everything into the integration by parts formula:

$$\int \underbrace{x}_{u} \underbrace{\sin x \, dx}_{dv} = \underbrace{x}_{u} \cdot \underbrace{(-\cos x)}_{v} - \int \underbrace{(-\cos x)}_{v} \underbrace{dx}_{du}$$

$$= -x\cos x + \int \cos x \, dx$$

$$= \boxed{-x\cos x + \sin x + C}$$

E.g. Find $\int \underbrace{x}_{u}\underbrace{e^{2x}dx}_{dv} = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C$

Let $\begin{cases} u = x \\ dv = e^{2x}dx \end{cases}$ Then $\begin{cases} du = dx \\ v = \int e^{2x}dx = \frac{e^{2x}}{2} \end{cases}$

Plug in the integration by parts formula.

$$\int \underbrace{x}_{u}\underbrace{e^{2x}dx}_{dv} = \underbrace{x}_{u} \cdot \underbrace{\frac{e^{2x}}{2}}_{v} - \int \underbrace{\frac{e^{2x}}{2}}_{v}\underbrace{dx}_{du}$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}dx$$

$$= \boxed{\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C}$$

E.g. $\int \underbrace{\ln(x)}_{u} \underbrace{dx}_{dv}$

Let $\begin{cases} u = \ln(x) \\ dv = dx \end{cases}$ Then $\begin{cases} du = \frac{1}{x}dx \\ v = \int dx = x \end{cases}$

By the integration by parts formula:

$$\int \ln(x)\,dx = x\ln(x) - \int \cancel{x}\cdot\frac{1}{\cancel{x}}\,dx$$

$$= x\ln(x) - \int 1\,dx$$

$$= \boxed{x\ln(x) - x + C}$$

* Shortcut to the Integration by Parts formula (Tabular method for Integration by parts)

E.g. $\int x \sin x\, dx.$ $\begin{cases} u = x \\ dv = \sin x\, dx \end{cases}$

u

D		I
x	$+$	$\sin x$
1	$-$	$-\cos x$
0		$-\sin x$

other function of the integrand

Answer: $\int x \sin x \, dx = + x \cdot (-\cos x) - (-\sin x)$

$$= \boxed{-x\cos x + \sin x + C}$$

E.g. $\int x^2 \cos x \, dx$

1st way: Integration-by-parts formula.

Let $\begin{cases} u = x^2 \\ dv = \cos x \, dx \end{cases}$ Then $\begin{cases} du = 2x \, dx \\ v = \int \cos x \, dx = \sin x \end{cases}$

$$\int x^2 \cos x \, dx = x^2 \sin x - 2 \underbrace{\int x \sin x \, dx}$$

$\begin{cases} u = x \\ dv = \sin x \, dx \end{cases}$ $\begin{cases} du = dx \\ v = -\cos x \end{cases}$

$$\int x^2 \cos x \, dx = x^2 \sin x - 2 \cdot \left[-x \cos x - \int (-\cos x) \, dx \right]$$

$$= x^2 \sin x + 2x \cos x - 2 \int \cos x \, dx$$

$$= \boxed{x^2 \sin x + 2x \cos x - 2 \sin x + C}$$

2nd way: Tabular method

u

	D		I
	x^2	$+$	$\cos x$
	$2x$	$-$	$\sin x$
	2	$+$	$-\cos x$
	0		$-\sin x$

Answer: $x^2 \sin x - 2x(-\cos x) + 2(-\sin x)$

$$= \boxed{x^2 \sin x + 2x \cos x - 2 \sin x + C}$$

E.g. $\int e^x \sin x \, dx$

Q: How do we choose u and dv?

L.I.A.T.E Rule

L: Logarithmic Functions (E.g. $\ln(\ )$; $\log(\ )$; ...)

I: Inverse Trig Functions (E.g. $\arcsin(\ )$; $\arccos(\ )$, ...)

A: Algebraic Functions (E.g. Any polynomial, $\sqrt{x}$, $\frac{1}{x}$, ...)

T: Trigonometric Functions (E.g. $\sin(\ )$; $\tan(\ )$; ...)

E: Exponential Functions (E.g. $e^{(\ )}$; $a^{(\ )}$, ...)

→ Whichever function is ahead in the list should be u.