3.2. Trigonometric Integrals

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Goal: Apply trig identities and u-substitution to find trig integrals.

Reminder of trig identities:

$$\sin^2 x + \cos^2 x = 1 \;\rightarrow\; \sin^2 x = 1 - \cos^2 x, \quad \cos^2 x = 1 - \sin^2 x$$

$$\sec^2 x - \tan^2 x = 1 \;\rightarrow\; \sec^2 x = 1 + \tan^2 x, \quad \tan^2 x = \sec^2 x - 1$$

* Power Reduction Formulas:

$$\boxed{\sin^2 x = \frac{1 - \cos(2x)}{2}; \quad \cos^2(x) = \frac{1 + \cos(2x)}{2}}$$

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Double-Angle Formulas:

$$\boxed{\begin{gathered}\cos(2x) = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \cos^2 x - \sin^2 x \\ \sin(2x) = 2\sin x \cos x\end{gathered}}$$

Differentiation.

$$\boxed{\begin{gathered}\frac{d}{dx}(\sin x) = \cos x \;;\; \frac{d}{dx}(\cos x) = -\sin x \\ \frac{d}{dx}(\tan x) = \sec^2 x \;;\; \frac{d}{dx}(\sec x) = \sec x \tan x\end{gathered}}$$

Integration:

$$\boxed{\begin{gathered}\int \tan x \, dx = \ln|\sec x| + C \\ \int \sec x \, dx = \ln|\sec x + \tan x| + C\end{gathered}}$$

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Trig Integrals:

E.g. Find $\int \cos^3 x \, dx$

Trig identity

$$= \int \cos^2 x \cdot \cos x \, dx = \int \Big(1 - \underbrace{\boxed{\sin^2 x}}_{u^2}\Big) \cdot \underbrace{\boxed{\cos x \, dx}}_{du}$$

Let $\boxed{u = \sin x}$. Then $du = \cos x \, dx$

$$= \int (1 - u^2) \, du = u - \frac{u^3}{3} + C$$

$$= \boxed{\sin x - \frac{\sin^3 x}{3} + C}$$

E.g. Find $\int \sin^5 x \cos^2 x \, dx$

$$= -\int \underbrace{\sin^4 x}\ \underbrace{\cos^2 x}_{u^2}\ \underbrace{(-\sin x\, dx)}_{du}$$

$$(\sin^2 x)^2 = (1-\cos^2 x)^2$$
$$= (1-u^2)^2$$

Let $u = \cos x$. Then $du = -\sin x\, dx$.

$$= -\int (1-u^2)^2 \cdot u^2\, du$$

$$= -\int (1-2u^2+u^4)\cdot u^2\, du$$

$$= -\int (u^2 - 2u^4 + u^6)\, du$$

$$= -\frac{u^3}{3} + \frac{2u^5}{5} - \frac{u^7}{7} + C$$

$$= \boxed{-\frac{\cos^3 x}{3} + \frac{2\cos^5 x}{5} - \frac{\cos^7 x}{7} + C}$$

E.g. Find $\int_0^{\pi} \sin^2 x \, dx$.

Power Reduction Formula

$$= \int_0^{\pi} \frac{1 - \cos(2x)}{2} dx = \frac{1}{2} \int_0^{\pi} \left(1 - \cos(2x)\right) dx$$

$$= \frac{1}{2}\left(x - \frac{\sin(2x)}{2}\right)\Big|_0^{\pi}$$

$$= \frac{1}{2}\left(\pi - \frac{\sin(2\pi)}{2}\right) = \boxed{\frac{\pi}{2}}$$

E.g. $\int \sin^4 x \, dx$

$$= \int (\sin^2 x)^2 \, dx = \int \left(\frac{1 - \cos(2x)}{2} \right)^2 dx$$

$$= \frac{1}{4} \int (1 - \cos(2x))^2 \, dx$$

$$= \frac{1}{4} \int \left[1 - 2\cos(2x) + \underbrace{\cos^2(2x)} \right] dx$$

$$\int \cos^2(2x) \, dx = \frac{1}{2} \int (1 + \cos(4x)) \, dx = \frac{1}{2} \left(x + \frac{\sin(4x)}{4} \right)$$

Power Reduction

$$= \frac{1}{4} \left[x - \sin(2x) + \frac{1}{2} \left(x + \frac{\sin(4x)}{4} \right) \right] + C$$

$$= \boxed{\frac{1}{4}\left[\frac{3}{2}x - \sin(2x) + \frac{\sin(4x)}{8}\right] + C}$$

Strategy for dealing with integrals of the form

$$\boxed{\int \sin^m x \cos^n x \, dx, \quad m, n \text{ are nonnegative integers}}$$

Scenario 1: Power of cosine is odd, i.e., n is odd, say $n = 2k+1$.

Save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factor as an expression in sine only. Then use the u-sub: $\boxed{u = \sin x}$

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \cos^{2k} x \cdot \boxed{\cos x \, dx} \to du$$

$$= \int \underbrace{\sin^m x}_{u^m} \cdot \left(1 - \underbrace{\sin^2 x}_{u^2}\right)^k \overbrace{\boxed{\cos x \, dx}}^{du}$$