$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\rightarrow y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$\rightarrow y = \pm b\sqrt{1 - \frac{x^2}{a^2}} = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$

By Symmetry, find area A_1 and multiply by 4

$$A_1 = \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx = \frac{b}{a}\cdot\int_0^a \boxed{\sqrt{a^2-x^2}}\,dx$$

Trig Sub: $\boxed{x = a\sin\theta}$; $\boxed{dx = a\cos\theta\, d\theta}$

Bounds for θ: $x = 0 \rightarrow a\sin\theta = 0 \rightarrow \sin\theta = 0$

$\rightarrow \boxed{\theta = 0}$

$x = a \rightarrow a\sin\theta = a \rightarrow \sin\theta = 1$

$\rightarrow \boxed{\theta = \frac{\pi}{2}}$

$$= \frac{b}{\cancel{a}} \int_0^{\pi/2} \cancel{a}\cos\theta \cdot a\cos\theta \, d\theta = ab \int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta$$

$$= ab \cdot \int_0^{\pi/2} \frac{1+\cos(2\theta)}{2} d\theta = \frac{ab}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$= \frac{ab}{2} \cdot \left(\frac{\pi}{2}\right) = \frac{\pi ab}{4}$$

$\longrightarrow$ Area of Ellipse : $\boxed{\pi ab}$

E.g. $$\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{\boxed{(4x^2+9)^{3/2}}} dx$$

$\to 27\sec^3\theta$

Let $x = \frac{3}{2}\tan\theta$

$dx = \frac{3}{2}\sec^2\theta \, d\theta$

Tuesday, October 2, 2018 9:23 AM

$$\left(4x^2+9\right)^{\frac{3}{2}} = \left(\sqrt{4x^2+9}\right)^3 = \left(\sqrt{4\cdot\left(\frac{9}{4}\tan^2\theta\right)+9}\right)^3$$

$$= \left(\sqrt{9\left(\tan^2\theta+1\right)}\right)^3 = 27\sec^3\theta$$

$$\int \frac{\frac{\cancel{27}}{8}\tan^3\theta}{\cancel{27}\sec^{\cancel{3}}\theta}\,\frac{3}{2}\cancel{\sec^2\theta}\,d\theta = \frac{3}{16}\int_0^{\frac{\pi}{3}} \frac{\tan^3\theta}{\sec\theta}\,d\theta$$

$$x=0 \rightarrow \frac{3}{2}\tan\theta = 0 \rightarrow \tan\theta = 0 \rightarrow \theta = 0$$

$$x = \frac{3\sqrt{3}}{2} \rightarrow \frac{3}{2}\tan\theta = \frac{3\sqrt{3}}{2} \rightarrow \tan\theta = \sqrt{3} \rightarrow \theta = \frac{\pi}{3}$$

$$\frac{3}{16}\int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta}\,d\theta = \frac{3}{16}\int_0^{\pi/3} \frac{\sin^3\theta}{\cos^2\theta}\,d\theta$$

$$= -\frac{3}{16}\int_0^{\pi/3} \frac{\sin^2\theta}{\cos^2\theta}\underbrace{(-\sin\theta\, d\theta)}_{du}$$

$\sin^2\theta \to 1-u^2$, $\cos^2\theta \to u^2$

Let $u = \cos\theta$; $du = -\sin\theta\, d\theta$

$\theta = 0 \to \cos(0) = 1$; $\theta = \frac{\pi}{3} \to \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$$= -\frac{3}{16}\int_1^{1/2} \frac{1-u^2}{u^2}\, du = \frac{3}{16}\int_{1/2}^{1} \frac{1-u^2}{u^2}\, du$$

$$= \frac{3}{16}\int_{1/2}^{1} \left(u^{-2} - 1\right) du$$

$$= \frac{3}{16}\left(-\frac{1}{u} - u\right)\Bigg|_{1/2}^{1} = \boxed{\frac{3}{32}}$$

E.g. $\displaystyle\int \frac{dx}{\sqrt{\underbrace{x^2+4x}\ -12}}$

* Complete the Square first.

$$\int \frac{dx}{\sqrt{\underbrace{x^2+4x+4}-16}}$$

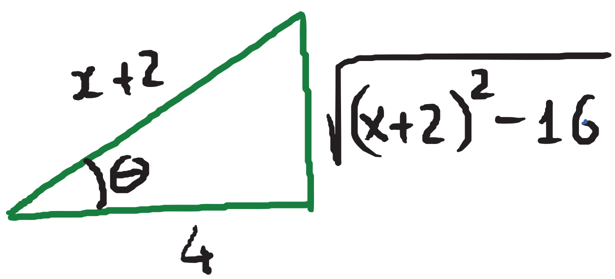
$$\int \frac{\boxed{dx}}{\sqrt{(x+2)^2-16}}$$

* Trig Sub: $x+2 = 4\sec\theta$; $\boxed{dx = 4\sec\theta\tan\theta\, d\theta}$

$$\int \frac{4\sec\theta\tan\theta}{\sqrt{16\sec^2\theta-16}}\, d\theta = \int \frac{\cancel{4}\sec\theta\,\cancel{\tan\theta}}{\cancel{4\tan\theta}}\, d\theta$$

$$= \int \sec\theta \, d\theta = \ln\left|\sec\theta + \tan\theta\right| + C$$

$$x + 2 = 4\sec\theta \rightarrow \sec\theta = \frac{x+2}{4}$$



$$\tan\theta = \frac{\sqrt{(x+2)^2 - 16}}{4}$$

$$= \boxed{\ln\left|\frac{x+2}{4} + \frac{\sqrt{(x+2)^2 - 16}}{4}\right| + C}$$

E.g.

$$\int \frac{x}{\sqrt{3 - 2x - x^2}} \, dx = \int \frac{\boxed{x}}{\sqrt{4 - (x+1)^2}} \, dx$$

Complete the square: $3 + 1 - (x^2 + 2x + 1)$

$$= 4 - (x+1)^2$$

$x+1 = 2\sin\theta$

$\rightarrow x = 2\sin\theta - 1$

Tuesday, October 2, 2018 10:02 AM

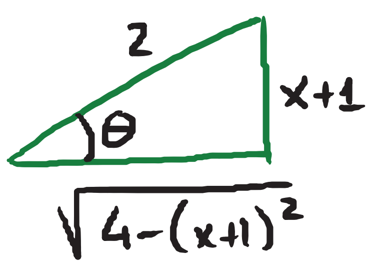
$\boxed{x+1 = 2\sin\theta} \longrightarrow dx = 2\cos\theta \, d\theta$

$$\sqrt{4-(x+1)^2} = \sqrt{4-4\sin^2\theta} = 2\cos\theta$$

$$\int \frac{2\sin\theta - 1}{2\cancel{\cos\theta}} \, 2\cancel{\cos\theta} \, d\theta$$

$$= \int (2\sin\theta - 1)\, d\theta = -2\boxed{\cos\theta} - \boxed{\theta} + C$$

$x+1 = 2\sin\theta \rightarrow \sin\theta = \dfrac{x+1}{2} \rightarrow \theta = \arcsin\left(\dfrac{x+1}{2}\right)$



$\rightarrow \cos\theta = \dfrac{\sqrt{4-(x+1)^2}}{2}$

Final answer: $\boxed{-\sqrt{4-(x+1)^2} - \arcsin\left(\frac{x+1}{2}\right) + C.}$