3.3. Trig Substitution

Tuesday, October 2, 2018 8:02 AM

Goal: Find integrals when the integrands contain radicals expressions by using trig substitution.

Strategy for trig substitution

Expression	Trig Sub	Identity
$\boxed{\sqrt{a^2 - x^2}}$	$\boxed{x = a\sin\theta}$ $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$\sqrt{a^2 - a^2\sin^2\theta}$ $= \sqrt{a^2(1 - \sin^2\theta)}$ $= \lvert a\rvert \sqrt{\cos^2\theta}$ $= \boxed{\lvert a\rvert \cdot \cos\theta}$
$\boxed{\sqrt{a^2 + x^2}}$	$x = a\tan\theta$ $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\sqrt{a^2 + a^2\tan^2\theta}$ $= \sqrt{a^2(1 + \tan^2\theta)}$ $= \lvert a\rvert \sqrt{\sec^2\theta}$ $= \boxed{\lvert a\rvert \sec\theta}$

$$\boxed{\sqrt{x^2 - a^2}}$$

$$\boxed{x = a\sec\theta}$$

$0 \le \theta < \frac{\pi}{2}$ or

$\pi \le \theta < \frac{3\pi}{2}$

$$\sqrt{a^2\sec^2\theta - a^2}$$
$$= \sqrt{a^2(\sec^2\theta - 1)}$$
$$= |a|\sqrt{\tan^2\theta}$$
$$= \boxed{|a|\tan\theta}$$

E.g. Find $\int \sqrt{9 - x^2}\, dx$

Trig Sub: $\boxed{x = 3\sin\theta}$; $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$dx = 3\cos\theta\, d\theta$$

Plug in: $\int \sqrt{9 - 9\sin^2\theta} \cdot 3\cos\theta\, d\theta$

$$= \int \sqrt{9(1 - \sin^2\theta)} \cdot 3\cos\theta\, d\theta$$

$$= \int 3 \cdot \cos\theta \cdot 3\cos\theta \, d\theta = 9 \int \cos^2\theta \, d\theta$$

Power Reduction

$$= \frac{9}{2} \int \left[1 + \cos(2\theta)\right] d\theta$$

$$= \frac{9}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right) + C$$

$$= \frac{9}{2}\theta + \frac{9}{4}\sin(2\theta) + C$$

* How do we get back to x

$$x = 3\sin\theta \longrightarrow \sin\theta = \frac{x}{3} \longrightarrow \theta = \arcsin\left(\frac{x}{3}\right)$$

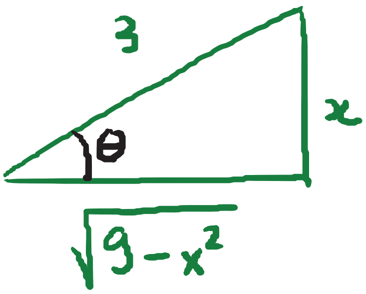
$$= \frac{9}{2}\arcsin\left(\frac{x}{3}\right) + \boxed{\frac{9}{4}\sin\Big(2\underbrace{\boxed{\arcsin\left(\frac{x}{3}\right)}}_{}\!\!\!\!\!\!\!\!\!\!\overset{\theta}{}\Big)} + C$$

further simplication

$$\sin(2\theta) = 2\,\boxed{\sin\theta}\,\cos\theta\cdot = \frac{2}{9}x\cdot\sqrt{9-x^2}$$

$$\frac{x}{3}$$

$$\boxed{\frac{x}{3} = \sin\theta}$$



$$\boxed{\cos\theta = \frac{\sqrt{9-x^2}}{3}}$$

Completely simplified answer:

$$\boxed{\frac{9}{2}\arcsin\left(\frac{x}{3}\right) + \frac{1}{2}x\cdot\sqrt{9-x^2} + C}$$

Process of Trig Sub:

1. Make the appropriate substitution
2. Get an integral in terms of θ $\rightarrow$ evaluate the integral.

(3) Use identities or geometry to express the answer in terms of x. (inverse trig (trig) can always be simplified)

E.g. $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$

E.g. $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}}$; a is a constant and $a>0$.

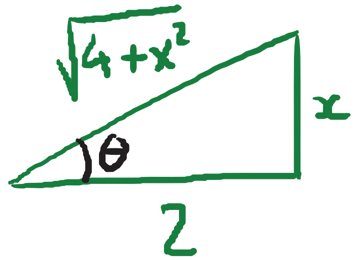
E.g. $\displaystyle\int \frac{dx}{\boxed{\sqrt{4+x^2}}}$ let $x = 2\tan\theta$; $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$dx = 2\sec^2\theta \, d\theta$

$$= \int \frac{\cancel{2}\,\sec^{\cancel{2}}\theta}{\boxed{\cancel{2}\,\cancel{\sec\theta}}}\, d\theta = \int \sec\theta \, d\theta$$

Formula $= \ln|\sec\theta + \tan\theta| + C$

$$x = 2\tan\theta \longrightarrow \tan\theta = \frac{x}{2} = \frac{\text{opp.}}{\text{adj.}}$$



$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{hyp.}}{\text{adj.}} = \frac{\sqrt{4+x^2}}{2}$$

<u>Final Answer:</u> $\boxed{\ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C}$

<u>E.g.</u> $$\int \frac{dx}{\sqrt{x^2 - a^2}}$$

Let $x = a\sec\theta$, $0 \le \theta < \frac{\pi}{2}$; $\pi \le \theta < \frac{3\pi}{2}$

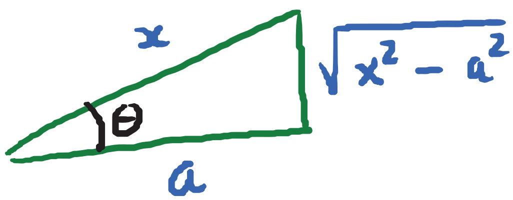
$$\boxed{dx = a\sec\theta\tan\theta\, d\theta}$$

$$\boxed{\sqrt{x^2 - a^2} = \sqrt{a^2(\sec^2\theta - 1)} = a\tan\theta}$$

$$\int \frac{\cancel{a}\sec\theta\,\cancel{\tan\theta}\, d\theta}{\cancel{a}\,\cancel{\tan\theta}} = \int \sec\theta\, d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C$$

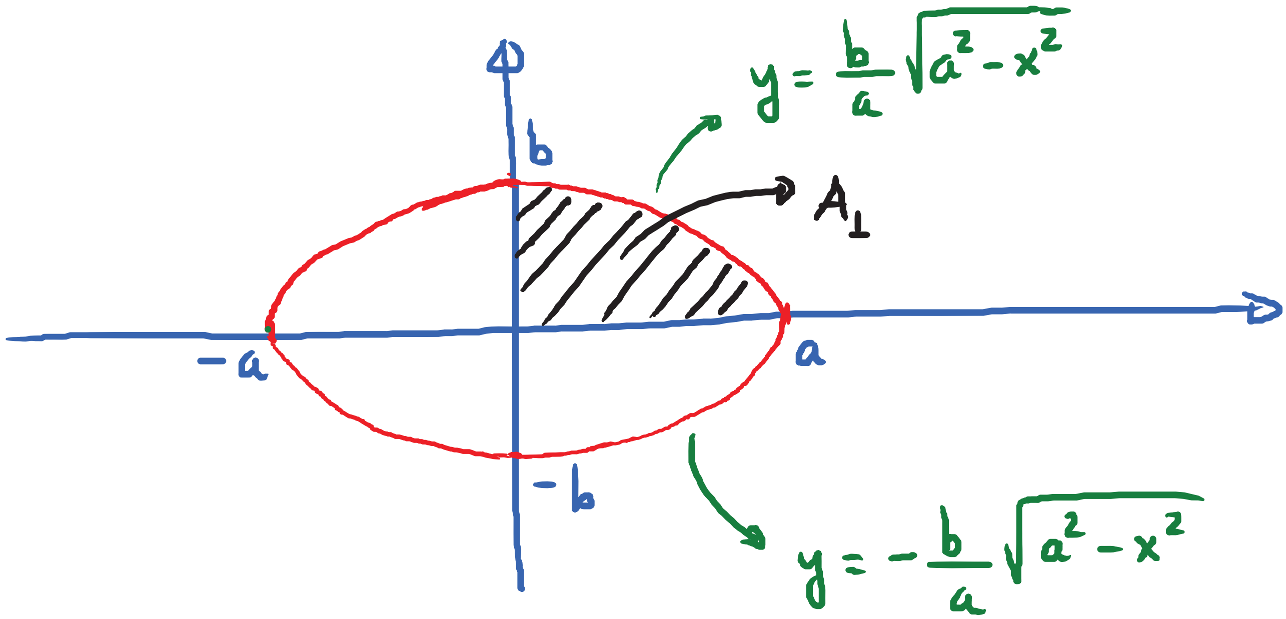
$$x = a\sec\theta \rightarrow \sec\theta = \frac{x}{a}$$



$$\rightarrow \tan\theta = \frac{\sqrt{x^2-a^2}}{a}$$

Answer: $\ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C$

E.g.



Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

Find the area of this ellipse.