Step 3: Integrate

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx = \frac{1}{2}\int \frac{dx}{x} + \frac{1}{5}\int \frac{dx}{2x-1} - \frac{1}{10}\int \frac{dx}{x+2}$$

$$= \frac{1}{2}\ln|x| + \frac{1}{5}\cdot\frac{1}{2}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C$$

$$= \boxed{\frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C}$$

Ex. Find $-\int \frac{-\sin(x)}{\cos^2 x + \cos(x) - 30}\,dx$ du

Let $u = \cos x$. Then $du = -\sin x\,dx$

$$-\int \frac{du}{u^2+u-30} = -\int \frac{du}{(u-5)(u+6)}$$

$\longrightarrow$ P.F.D. $-\left(\frac{1}{11}\int\frac{du}{u-5}-\frac{1}{11}\int\frac{du}{u+6}\right)$

$$= -\frac{1}{11}\ln|u-5| + \frac{1}{11}\ln|u+6| + C$$

$$= -\frac{1}{11}\ln|\cos x-5| + \frac{1}{11}\ln|\cos x+6| + C$$

Scenario 2: $Q(x)$ is a product of linear factors, some of which are repeated.

$$Q(x) = \boxed{(a_1x+b_1)^n}(a_2x+b_2)\cdots(a_mx+b_m)$$

(Say, the first factor is repeated n times)

Form of P.F.D.

$$\frac{P(x)}{Q(x)} = \boxed{\frac{\alpha_1}{a_1x+b_1}+\frac{\alpha_2}{(a_1x+b_1)^2}+\cdots+\frac{\alpha_n}{(a_1x+b_1)^n}} + \frac{A_2}{a_2x+b_2} + \cdots$$

E.g. Find $\displaystyle\int \frac{4x}{(x-1)^2(x+1)}\,dx$

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

Result: $A = 1$; $B = 2$; $C = -1$

$$\int \frac{4x}{(x-1)^2(x+1)}\,dx = \int \frac{dx}{x-1} + 2\int \frac{dx}{(x-1)^2} - \int \frac{dx}{x+1}$$

$$= \ln|x-1| + 2\int (x-1)^{-2}\,dx - \ln|x+1|$$

$$= \ln|x-1| + 2\cdot\frac{(x-1)^{-1}}{-1} - \ln|x+1|$$

$$= \boxed{\ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C}$$

Scenario 3: $Q(x)$ contains irreducible quadratic factors, none of which is repeated.

Note: A quadratic factor ax^2+bx+c is irreducible (cannot be factored over the real) if $b^2-4ac<0$

Suppose that $Q(x)$ contains an irreducible quadratic factor ax^2+bx+c, then in the P.F.D. we will have the term:

$$\frac{Ax+B}{ax^2+bx+c}$$

E.g. $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx$

$$= \int \frac{2x^2 - x + 4}{x\underbrace{(x^2 + 4)}_{\text{irreducible}}}\,dx$$

P.F.D.

$$\frac{2x^2 - x + 4}{x(x^2 + 4)} = \frac{Ax + B}{x^2 + 4} + \frac{C}{x}$$

Find A, B, and C.

$C = 1$; $A = 1$; $B = -1$

$$\int \frac{2x^2 - x + 4}{x(x^2+4)}\,dx = \int \frac{x-1}{x^2+4}\,dx + \int \frac{dx}{x}$$

$du = 2x\,dx$

$$= \frac{1}{2}\int \frac{2x}{\underbrace{x^2+4}_{u}}\,dx - \int \frac{dx}{x^2+4} + \int \frac{dx}{x}$$

$$= \boxed{\frac{1}{2}\ln|x^2+4| - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + \ln|x| + C}$$

E.g. $\displaystyle\int \frac{dx}{x^2+4x+18}$

$\displaystyle\int \frac{du}{u^2+14}$

Complete the Square

$$\int \frac{dx}{x^2+4x+4+14} = \int \frac{dx}{(\underbrace{x+2}_{u})^2 + 14}$$

$$= \frac{1}{\sqrt{14}}\arctan\left(\frac{x+2}{\sqrt{14}}\right) + C$$

Scenario 4: $Q(x)$ contains Repeated Irreducible quadratic factors.

If $Q(x)$ has a factor of the form $(ax^2+bx+c)^n$ where ax^2+bx+c is irreducible, then in the P.F.D., we will have:

$$\frac{\alpha_1 x+\beta_1}{ax^2+bx+c}+\frac{\alpha_2 x+\beta_2}{(ax^2+bx+c)^2}+\cdots+\frac{\alpha_n x+\beta_n}{(ax^2+bx+c)^n}$$