3.4 Integration of Rational Functions by Partial Fraction Decomposition

Thursday, October 4, 2018 8:06 AM

E.g. Find $\int \frac{5}{x^2+x-2}\,dx$

$$= \int \boxed{\frac{5}{(x-1)(x+2)}}\,dx$$

→ Partial Fraction Decomposition.

$$\left.\frac{5}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}\right\}$$

Goal: find A and B so that LHS = RHS regardless of the value of x

Multiply both sides by $(x-1)(x+2)$

$$5 = A(x+2) + B(x-1)$$

$$5 = Ax + 2A + Bx - B$$

$$5 = \boxed{(A+B)}x + \boxed{2A-B}$$

$$\boxed{0} \cdot x + \boxed{5}$$

$$\longrightarrow \begin{cases} A + B = 0 \\ 2A - B = 5 \end{cases}$$

TI-84: $\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & -1 & 5 \end{array}\right)$

$\boxed{2^{nd}} \longrightarrow \overset{\text{Matrix}}{\boxed{x^{-1}}} \longrightarrow$ Edit $\longrightarrow$ Enter the matrix $\longrightarrow$ Quit

$\boxed{2^{nd}} \longrightarrow \overset{\text{Matrix}}{\boxed{x^{-1}}} \longrightarrow$ Math $\longrightarrow$ B: RREF (of the matrix)

$$\left(\begin{array}{cc|c} 1 & 0 & \frac{5}{3} \\ 0 & 1 & -\frac{5}{3} \end{array}\right)$$

$$\boxed{A = \frac{5}{3} \; ; \; B = -\frac{5}{3}}$$

$$\frac{5}{(x-1)(x+2)} = \boxed{\frac{5/3}{x-1} - \frac{5/3}{x+2}} \rightarrow \text{P.F.D. of } \frac{5}{(x-1)(x+2)}$$

$$\rightarrow \int \frac{5}{(x-1)(x+2)}\,dx = \frac{5}{3}\int \frac{dx}{x-1} - \frac{5}{3}\int \frac{dx}{x+2}$$

$$= \boxed{\frac{5}{3}\ln|x-1| - \frac{5}{3}\ln|x+2| + C}$$

Reminder: $\int \frac{dx}{x} = \ln|x| + C$; $\int \frac{dx}{x \pm a} = \ln|x \pm a| + C$

$$\int \frac{dx}{1+x^2} = \arctan(x) + C \; ; \; \int \frac{du}{1+u^2} = \arctan(u) + C$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \; ; \; \int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$$

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→ How to integrate $\displaystyle\int \frac{P(x)}{Q(x)}\,dx$.

1st thing to do when dealing with these integrals is:

If $\deg P(x) \geq \deg Q(x)$, then we do long division to simplify the expression to the form where deg top < deg bottom.

2nd thing: Partial Fraction Decomposition (if necessary)

E.g. $\displaystyle\int \frac{x^2+3x+5}{x+1}\,dx$ deg top $= 2 >$ deg bottom $= 1$.

* Long Division:

$$
\begin{array}{r|l}
 & \boxed{x+2} \\
\hline
\boxed{x}+1 & \boxed{x^2}+3x+5 \\
 & -(x^2+x) \\
\hline
 & \boxed{2x}+5 \\
 & -(2x+2) \\
\hline
 & \boxed{3}
\end{array}
$$

$x+2$ → quotient

3 → Remainder

$\dfrac{x^2}{x} = x$

$\dfrac{2x}{x} = 2$

$$\frac{x^2+3x+5}{x+1} = \overset{Q}{\boxed{x+2}} + \frac{\overset{R}{\boxed{3}}}{\boxed{x+1}}$$

$$\int \frac{x^2+3x+5}{x+1}\,dx = \int (x+2)\,dx + 3\int \frac{dx}{x+1}$$

$$= \boxed{\frac{x^2}{2} + 2x + 3\ln|x+1| + C}$$

* Synthetic Division

(Note: It only works if the thing you divide by has the form $x+a$ or $x-a$)

$$\frac{x^2+3x+5}{x+1}$$

$$\begin{array}{r|rrr} & 1 & 3 & 5 \\ -1 & & -1 & -2 \\ \hline & 1 & 2 & \boxed{3} \end{array}$$

mult.

add

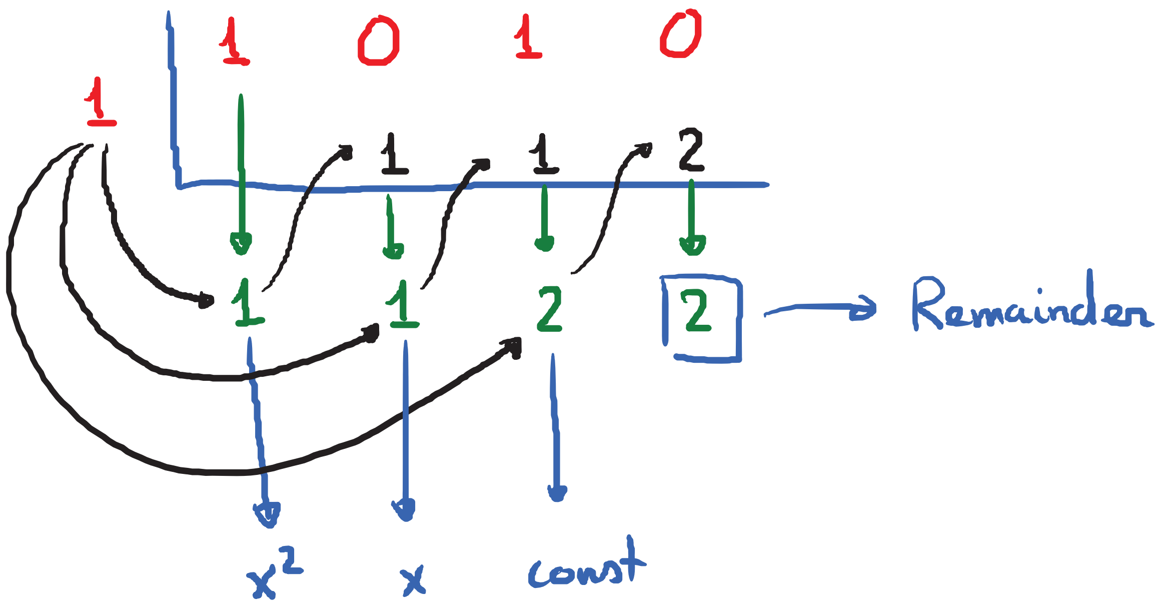
add

$\boxed{3}$ → remainder

$$\underbrace{\underset{x}{1} \qquad \underset{\text{const.}}{2}}_{\text{quotient}}$$

quotient : $\boxed{x+2}$

E.g. $\displaystyle\int \frac{x^3 + x}{x - 1}\,dx$



Quotient: $\boxed{x^2 + x + 2}$

$$\int \frac{x^3+x}{x-1}\,dx = \int\left(x^2 + x + 2 + \frac{2}{x-1}\right)dx$$

$$= \boxed{\frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C}$$

Now, assume $\deg P(x) < \deg Q(x)$

$\longrightarrow$ Find $\displaystyle\int \frac{P(x)}{Q(x)}\,dx$

$\longrightarrow$ Factor $Q(x)$ completely.

After we factor $Q(x)$ completely, there are a few scenarios.

Scenario 1: $Q(x)$ can be factored into a product of distinct linear factors

$$Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_mx+b_m)$$

P.F.D. (Form of P.F.D.)

$$\boxed{\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_m}{a_mx+b_m}}$$

E.g. Find $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}$

Step 1: Factor $Q(x)$ completely.

$$2x^3+3x^2-2x = x(2x^2+3x-2)$$
$$= \underbrace{x(2x-1)(x+2)}_{\text{3 distinct linear factors}}$$

Step 2: P.F.D.

$$\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

Find A, B and C

$$x^2+2x-1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$
$$x^2+2x-1 = A(2x^2+3x-2) + B(x^2+2x) + C(2x^2-x)$$
$$1x^2+\boxed{2}x-1 = \boxed{(2A+B+2C)}x^2 + \boxed{(3A+2B-C)}x - 2A$$

$$\begin{aligned} 2A + B + 2C &= 1 \\ 3A + 2B - C &= 2 \\ -2A &= -1 \end{aligned} \rightarrow \left(\begin{array}{ccc|c} 2 & 1 & 2 & 1 \\ 3 & 2 & -1 & 2 \\ -2 & 0 & 0 & -1 \end{array}\right)$$

$$\rightarrow A = \frac{1}{2};\ B = \frac{1}{5};\ C = -\frac{1}{10}.$$

2nd way to solve for A, B, C. Strategic Substitution

$$x^2 + 2x - 1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$

Plug $x = -2$ to both sides:

$$-1 = C \cdot (-2) \cdot (-5) \rightarrow -1 = 10C \rightarrow \boxed{C = -\frac{1}{10}}$$

Plug $x = 0$ to both sides:

$$-1 = A \cdot (-1) \cdot 2 \rightarrow -1 = -2A \rightarrow \boxed{A = \frac{1}{2}}$$

Plug $x = \frac{1}{2}$ to both sides: $\frac{1}{4} + 1 - 1 = B \cdot \left(\frac{1}{2}\right) \cdot \left(\frac{5}{2}\right)$

$$\rightarrow \boxed{B = \frac{1}{5}}$$