$$\boxed{\left|E_{T_n}\right| \le \underbrace{\frac{K(b-a)^3}{12n^2}}_{\text{upper bound for error}}}$$

where K is a number such that $|f''(x)| \le K$ for all x in $[a,b]$; i.e.; K is an upper bound for $|f''(x)|$ on $[a,b]$.

(In practice, we can take K to be the maximum of $|f''(x)|$ on $[a,b]$)

For Simpson's rule, the error bound formula is:

$$\boxed{\left|E_{S_n}\right| \leq \frac{K(b-a)^5}{180n^4}}$$

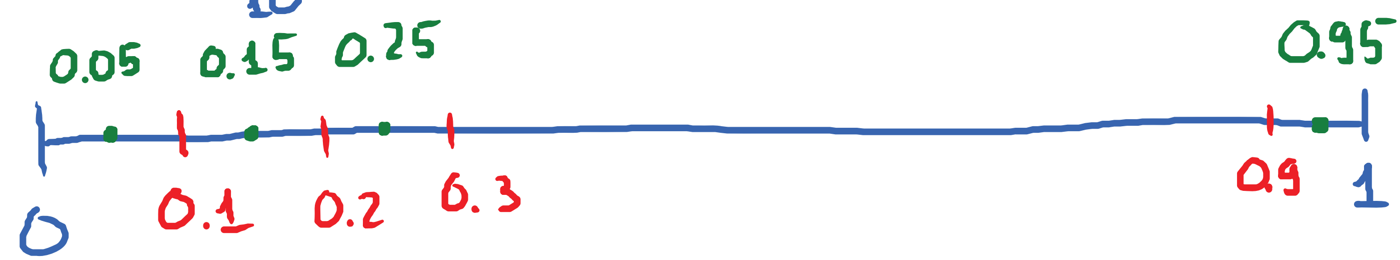
where K is a number such that $\left|f^{(4)}(x)\right| \leq K$ for all x in $[a,b]$. (In practice, we can just take $K = \max_{[a,b]} \left|f^{(4)}(x)\right|$)

E.g. Estimate $\int_0^1 e^{x^2}\,dx$.

1. Use the midpoint rule for $n=10$ to estimate this integral $\longrightarrow M_{10}$.

(2) Find an upper bound for the error when using M_{10} to estimate the integral; i.e., find an upper bound for $E_{M_{10}}$.

(1)



$$\Delta x = \frac{1}{10} = 0.1$$

$$M_{10} = 0.1 \cdot \left[f(0.05) + \cdots + f(0.95) \right]$$

$$M_{10} \approx 1.460393$$

(2) Find an upper bound for $E_{M_{10}}$.

By the upper bound formula for Midpoint :

$$\left| E_{M_{10}} \right| \leq \frac{K \cdot (b - a)^3}{24 n^2}$$

1 0 3

10

$b = 1$; $a = 0$; $n = 10$.

We need K. We can take K to be the maximum value of $|f''(x)|$ on $[0, 1]$.

$f(x) = e^{x^2}$; $f'(x) = 2xe^{x^2}$ (Chain Rule)

$$f''(x) = 2\left[1 \cdot e^{x^2} + x \cdot 2x \cdot e^{x^2}\right]$$

product rule

$$f''(x) = 2e^{x^2}\left(1 + 2x^2\right)$$

Goal: To find K, we need to find the max value

of $f''(x) = \underbrace{2e^{x^2}(1+2x^2)}$ on $[0,1]$

already positive, we don't need $|\cdot|$

Since f'' is an increasing function on $[0,1]$, to find the maximum value, we just need to plug in the right endpoint.

So, $\max\limits_{[0,1]} f''(x) = f''(1) = 2e\cdot(3) = 6e.$

So, $|E_{M_{10}}| \leq \dfrac{6e\cdot(1)^3}{24\cdot(10)^2} \approx \boxed{0.0067}$

error of your approximation cannot exceed this number.

E.g. Using the formula for the bound for the error in Simpson's method to find the smallest value of n such that the Simpson approximation S_n for the integral $\int_1^2 \frac{1}{x}\,dx$ is accurate to within 0.0001?