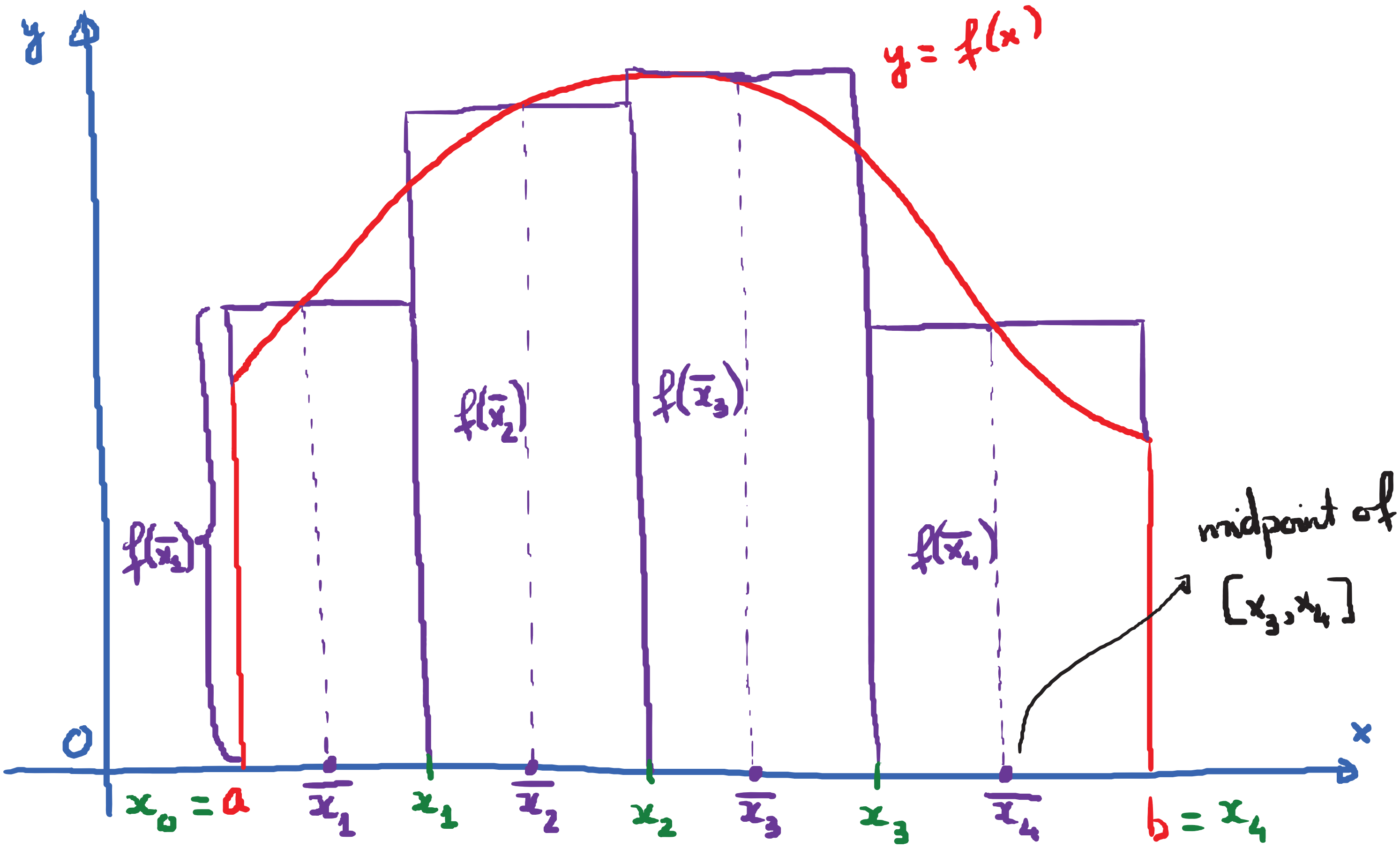
3.6. Numerical Integration

Tuesday, October 9, 2018 8:02 AM

Goals:
1. Midpoint Rule
2. Trapezoid Rule
3. Simpson's Rule
4. Error Estimates for these rules

1 Midpoint Rule

Estimate the integral $\int_a^b f(x)\,dx$; where $f(x) \geq 0$



$\int_a^b f(x)\,dx$ = area under the curve $y = f(x)$ from $x=a$ to $x=b$.

$\approx$ Sum of the areas of the rectangles

In this picture, we will denote the sum of the areas of these 4 rectangles by $\boxed{M}_{\boxed{4}}$

Midpoint

4 rectangles (divide $[a,b]$ into 4 sub-intervals)

Width of each rectangle in M_4 is $\frac{b-a}{4}$ ← call it Δx

The heights of the rectangles are: $f(\bar{x}_1); f(\bar{x}_2); f(\bar{x}_3); f(\bar{x}_4)$

Hence, $M_4 = f(\bar{x}_1)\cdot\Delta x + f(\bar{x}_2)\cdot\Delta x + f(\bar{x}_3)\cdot\Delta x + f(\bar{x}_4)\cdot\Delta x$

$$\longrightarrow \boxed{M_4 = \Delta x \cdot \sum_{i=1}^{4} f(\bar{x}_i)}$$

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E.g. $\displaystyle\int_0^1 \ln(x+5)\,dx$

* Find the exact result. Integration by parts:

$$\begin{cases} u = \ln(x+5) \\ dv = dx \end{cases} \rightarrow \begin{cases} du = \dfrac{1}{x+5}\,dx \\ v = x \end{cases}$$

$$x\ln(x+5)\Big|_0^1 - \int_0^1 \frac{x}{x+5}\,dx$$

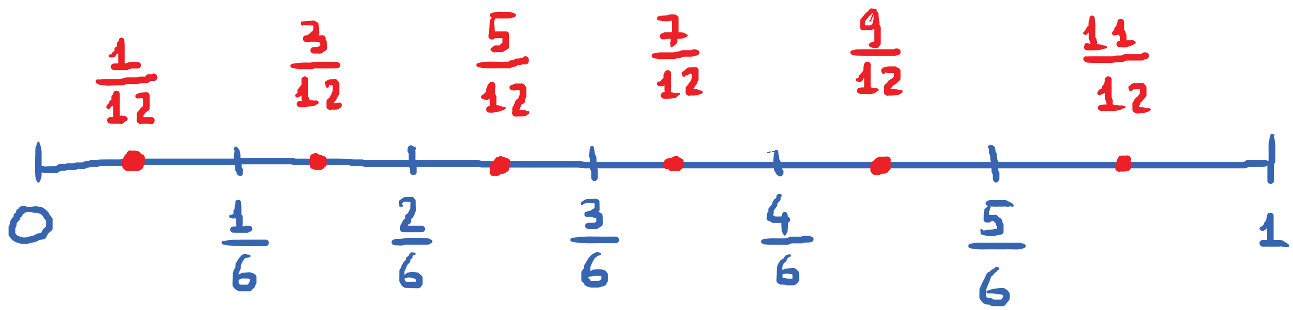
$$= \ln(6) - \int_0^1 \frac{(x+5)-5}{x+5}\,dx = \ln(6) - \int_0^1 \left(1 - \frac{5}{x+5}\right)dx$$

$$= \ln(6) - \left(x - 5\ln(x+5)\right)\Big|_0^1 = \ln(6) - \left[\left(1 - 5\ln(6)\right) + 5\ln(5)\right]$$

$$= \boxed{6\ln(6) - 1 - 5\ln(5)} \longrightarrow \text{exact result!}$$

* Use the Midpoint Rule to approximate this integral with $n = 6$.

$$\int_0^1 \ln(x+5)\,dx$$



$$\Delta x = \frac{1}{6}.$$

$$M_6 = \left[f\left(\frac{1}{12}\right) + f\left(\frac{3}{12}\right) + f\left(\frac{5}{12}\right) + \cdots + f\left(\frac{11}{12}\right) \right] \cdot \frac{1}{6}$$

$$M_6 \approx 1.7034$$

Process and Formula for the midpoint rule.

Estimate $\int_a^b f(x)\,dx$; assume $f(x) \geq 0$. Use n subintervals

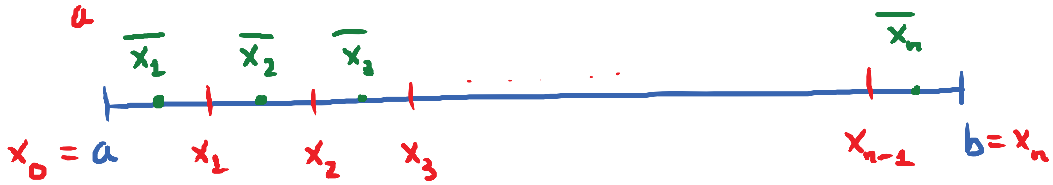
Step 1: Subdivide $[a, b]$ into n subintervals of equal length.

The length of each subinterval:

$$\Delta x = \frac{b-a}{n}.$$

Find the midpoints $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$ of the subintervals

$[\underbrace{x_0}_{a}, x_1]; [x_1, x_2], \dots, [x_{n-1}, \underbrace{x_n}_{b}]$

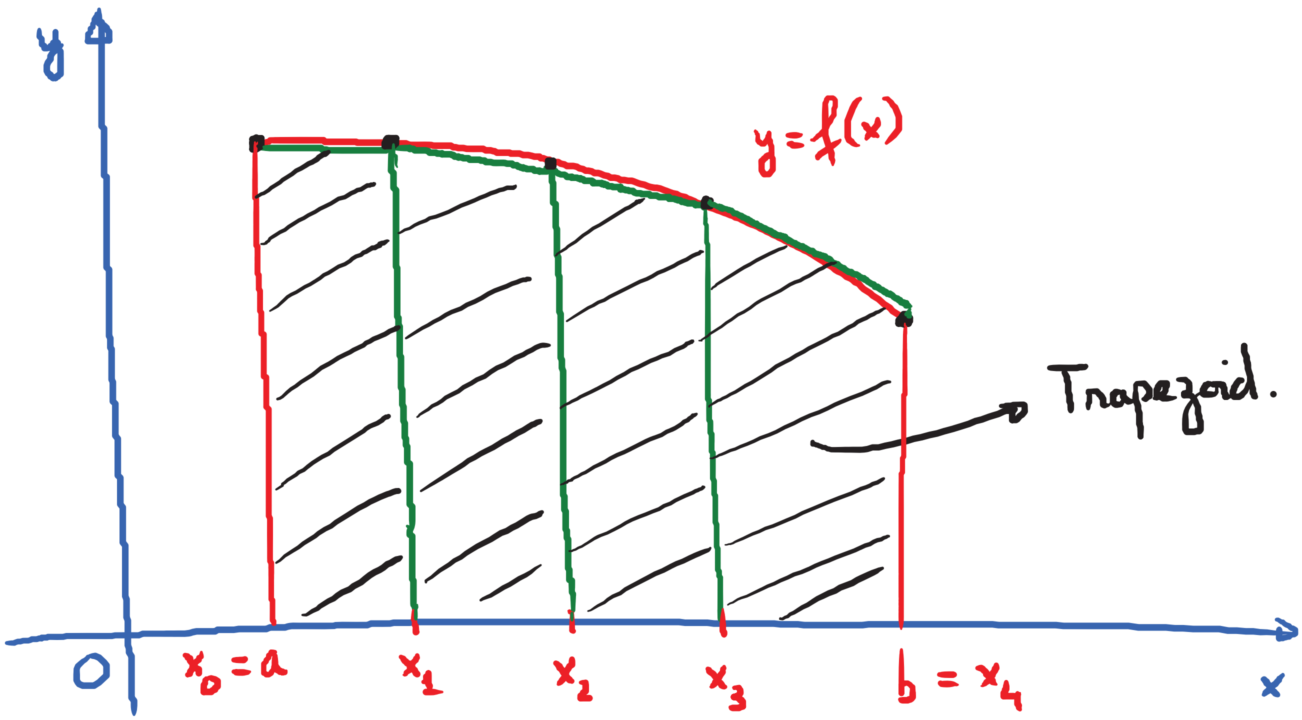


Step 2: Find the sum:

n^{th} midpt approx.

$$\boxed{M_n} = \Delta x \cdot \sum_{i=1}^{n} f(\bar{x}_i)$$

② Trapezoid Rule



1^{st} Trapezoid. Area $A_1 = \dfrac{[f(x_0)+f(x_1)]\,\Delta x}{2}$

Sum of the areas of these 4 trapezoids:

$$T_4 = \frac{\Delta x}{2}\Big[\big(f(x_0)+f(x_1)\big)+\big(f(x_1)+f(x_2)\big)+\big(f(x_2)+f(x_3)\big)+\big(f(x_3)+f(x_4)\big)\Big]$$