3.7 Improper Integrals

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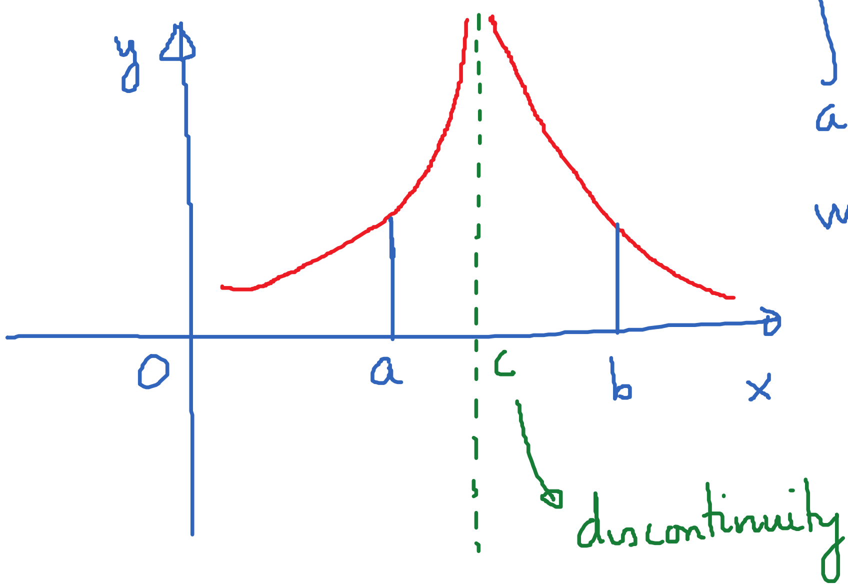
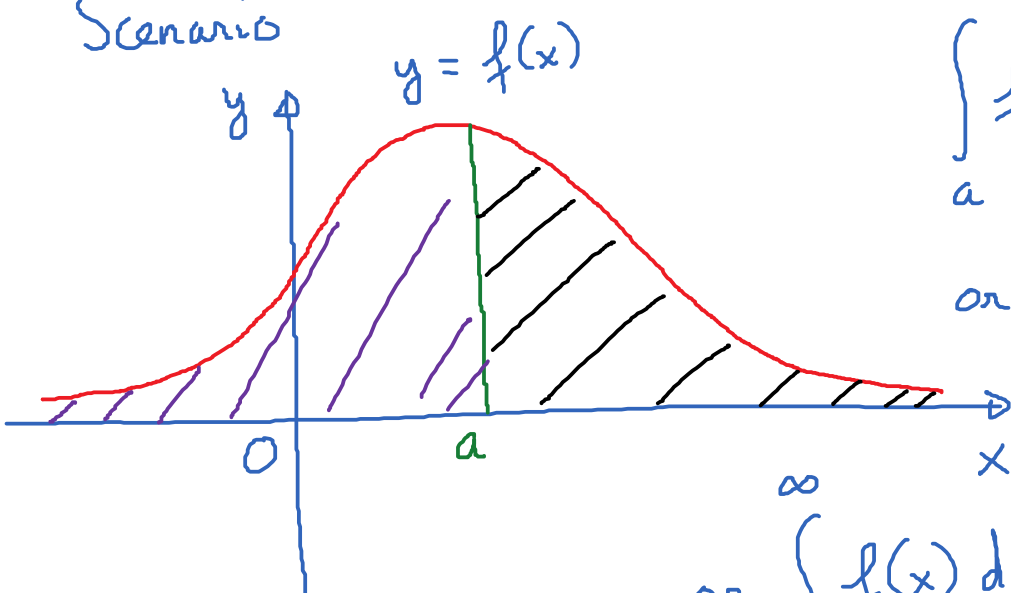
Scenario 1:



$$\int_a^b f(x)\,dx$$

where f has a discontinuity on $[a, b]$

Scenario



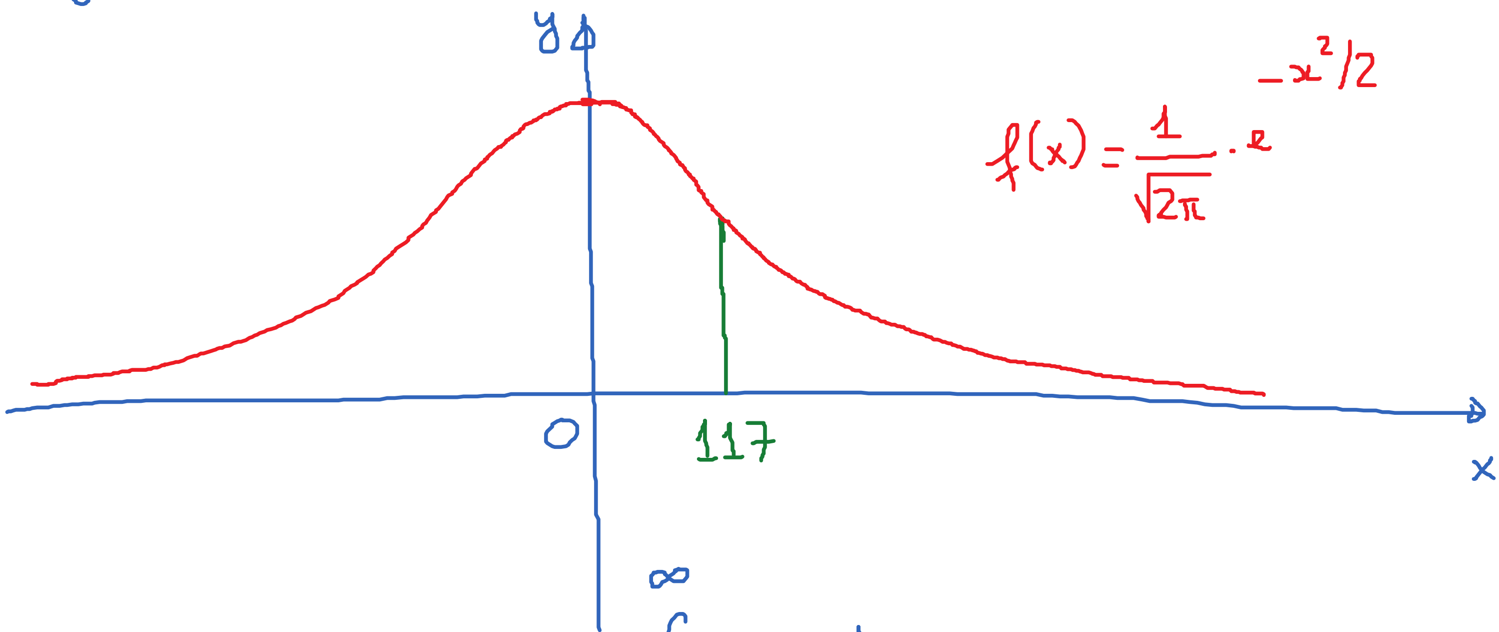
$$\int_a^\infty f(x)\,dx$$

or $$\int_{-\infty}^a f(x)\,dx$$

or $$\int_{-\infty}^\infty f(x)\,dx$$

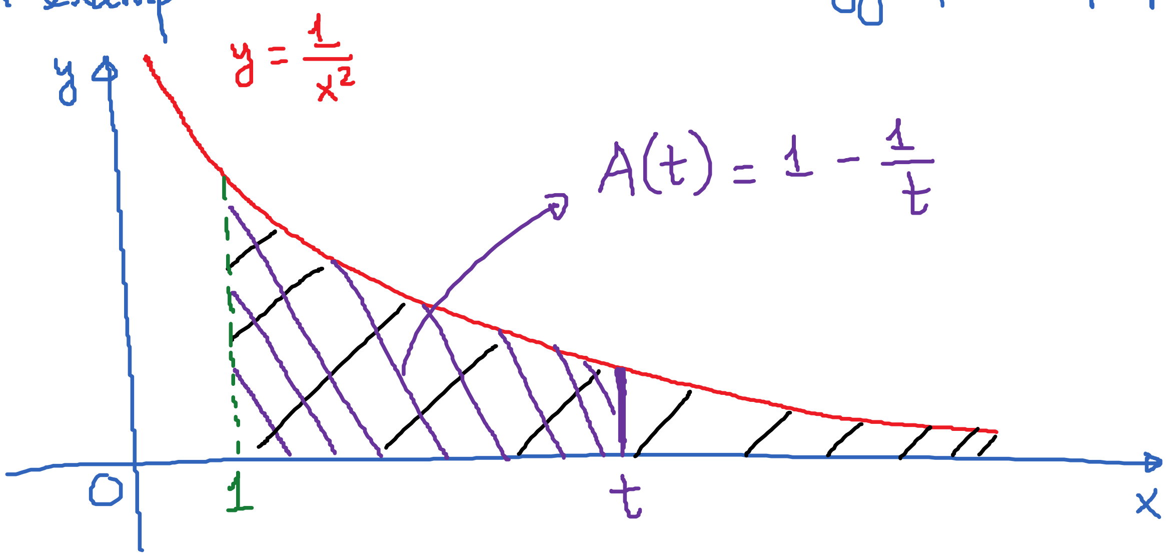
E.g Data is normally distributed

$$f(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$$



$$\%\text{ of data} > 117 = \int_{117}^{\infty} f(x)\,dx$$

An example to illustrate the strategy for improper integral



Find the area under the curve $y = \frac{1}{x^2}$ for $x > 1$.

Idea: first find the area under the curve from $x = 1$ to $x = t$ where $t > 1$. Then find the limit of the result as $t \to \infty$

$$\text{Area from } x=1 \text{ to } x=t = A(t) = \int_1^t \frac{1}{x^2}\,dx$$

$$A(t) = \int_1^t \frac{1}{x^2}\,dx = \int_1^t x^{-2}\,dx = \frac{x^{-1}}{-1}\Bigg|_1^t$$

$$= \left(-\frac{1}{x}\right)\Bigg|_1^t = -\frac{1}{t} + 1$$

$$A(t) = 1 - \frac{1}{t}.$$

Area under the curve from 1 to ∞ is

$$A = \lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = \boxed{1}$$

We just calculated an improper integral:

$$\boxed{\int_1^\infty \frac{1}{x^2}\,dx = 1}$$

Technique for finding Type I improper integrals:

Integrals of the form:

$$\int_a^\infty f(x)\,dx \quad \text{or} \quad \int_{-\infty}^a f(x)\,dx \quad \text{or} \quad \int_{-\infty}^\infty f(x)\,dx$$

(Bound(s) are not finite; but function continuous)

$$\boxed{\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx}$$

If the limit exists and is a finite #, we say that the integral converges to that #.

If the limit does not exist or infinite, we say that the integral diverges

Similarly,

$$\boxed{\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx}$$

Finally,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$$

where a is a real #.

We find the integrals on the RHS separately using the above formulas.

If both integrals are finite, the original integral converges.

If either one of the integral is infinite or does not exist, we say that the original integral diverges.

Ex.

1. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$

2. $\displaystyle\int_{-\infty}^0 xe^x\,dx$

3. $\displaystyle\int_{-\infty}^\infty xe^{-x^2}\,dx$

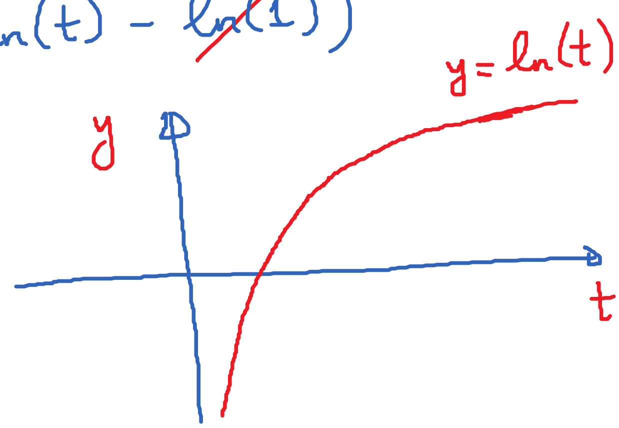
4. $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$

Determine whether the integral converges or diverges. If it converges, find the answer.

1. By definition, $\displaystyle\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \boxed{\int_1^t \frac{1}{x}\,dx}$

$$= \lim_{t\to\infty} \left(\ln x\right)\Big|_1^t = \lim_{t\to\infty}\left(\ln(t) - \cancelto{0}{\ln(1)}\right)$$

$$= \lim_{t\to\infty}\left(\ln(t)\right) = \infty$$



Conclusion: Integral diverges.

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(2) $$\int_{-\infty}^{0} x e^{x}\,dx = \lim_{t \to -\infty} \left(\boxed{\int_{t}^{0} x e^{x}\,dx} \right)$$

$$\int_{t}^{0} x e^{x}\,dx = x e^{x} \Big|_{t}^{0} - \int_{t}^{0} e^{x}\,dx = -t e^{t} - 1 + e^{t}$$

$$= \boxed{e^{t}(1-t) - 1}$$

$$\begin{cases} u = x \\ dv = e^{x}\,dx \end{cases} \longrightarrow \begin{cases} du = dx \\ v = e^{x} \end{cases}$$

$$\int_{-\infty}^{0} x e^{x}\,dx = \lim_{t \to -\infty} \left[\boxed{e^{t}(1-t)} - 1 \right]$$

$$= \lim_{t \to -\infty} \frac{1-t}{e^{-t}} - 1$$

L'Hopital

$$= \lim_{t \to -\infty} \boxed{\frac{-1}{\boxed{-e^{-t}}}} - 1 = \boxed{-1}$$

∞

$-\infty$

0

(3) $$\int_{-\infty}^{\infty} x e^{-x^2}\,dx = \int_{-\infty}^{0} x e^{-x^2}\,dx + \int_{0}^{\infty} x e^{-x^2}\,dx$$

$$= \lim_{t\to-\infty}\left(\int_{t}^{0} x e^{-x^2}\,dx\right) + \lim_{t\to\infty}\left(\int_{0}^{t} x e^{-x^2}\,dx\right)$$

$$\int x e^{-x^2}\,dx$$ Let $u = -x^2$; $du = -2x\,dx$

$$-\frac{1}{2}\int e^{u}\,du = -\frac{1}{2}e^{u} = \boxed{-\frac{1}{2}e^{-x^2}}$$

$$\int_{-\infty}^{\infty} x e^{-x^2}\,dx = \lim_{t\to-\infty}\left(-\frac{1}{2}e^{-x^2}\Big|_{t}^{0}\right) + \lim_{t\to\infty}\left(-\frac{1}{2}e^{-x^2}\Big|_{0}^{t}\right)$$

$$= \underbrace{\lim_{t\to-\infty}\left(-\frac{1}{2} + \underbrace{\boxed{\frac{1}{2}e^{-t^2}}}_{0}\right)}_{-\frac{1}{2}} + \underbrace{\lim_{t\to\infty}\left(\underbrace{\boxed{-\frac{1}{2}e^{-t^2}}}_{0} + \frac{1}{2}\right)}_{\frac{1}{2}}$$

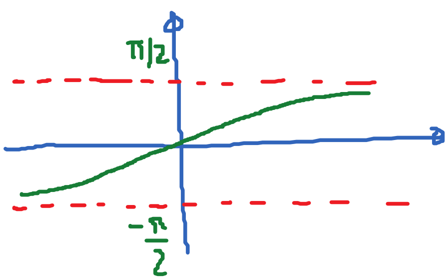
Since both limits are finite, the integral converges and it converges to $-\frac{1}{2}+\frac{1}{2}=0$

④ $$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx = \int_{-\infty}^{0}\frac{1}{1+x^2}\,dx + \int_{0}^{\infty}\frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to-\infty}\left(\int_{t}^{0}\frac{1}{1+x^2}\,dx\right) + \lim_{t\to\infty}\left(\int_{0}^{t}\frac{1}{1+x^2}\,dx\right)$$

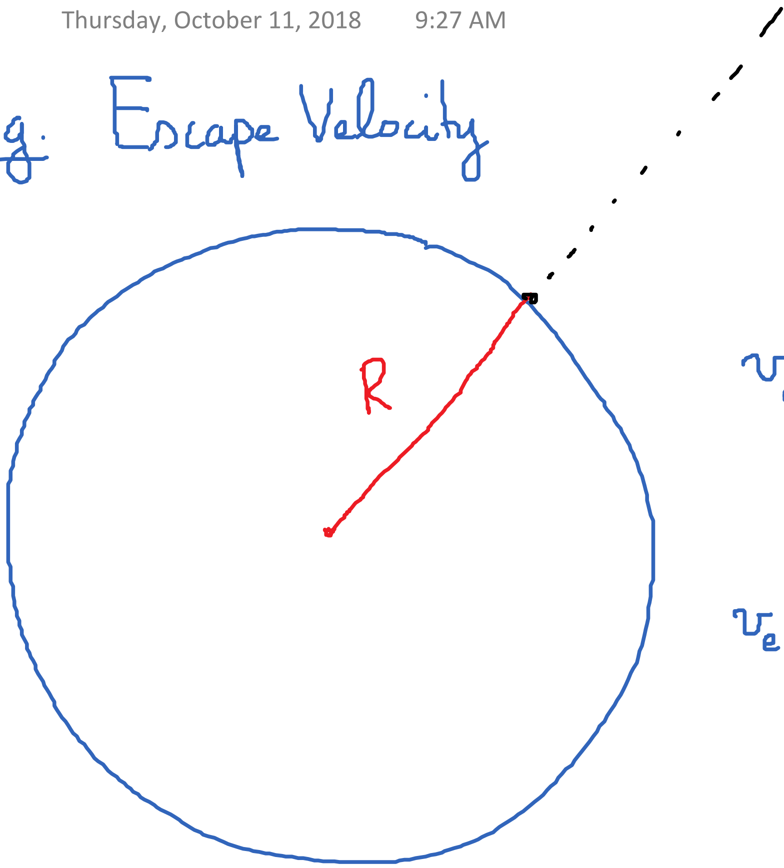
$$= \lim_{t\to-\infty}\left(\arctan(x)\Big|_{t}^{0}\right) + \lim_{t\to\infty}\left(\arctan(x)\Big|_{0}^{t}\right)$$

$$= \underbrace{\lim_{t\to-\infty}\left(\overset{0}{\cancel{\arctan(0)}} - \overset{(-\frac{\pi}{2})}{\arctan(t)}\right)}_{\frac{\pi}{2}} + \underbrace{\lim_{t\to\infty}\left(\overset{\frac{\pi}{2}}{\arctan(t)} - \overset{0}{\cancel{\arctan(0)}}\right)}_{\frac{\pi}{2}}$$



$$\frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

E.g. Escape Velocity



$$v_e^2 = 2\int_R^{\infty} \frac{GM}{x^2}\,dx$$

$$v_e = \sqrt{\frac{2GM}{R}}$$

E.g. p-integral.

For what values of p is the integral $\int_1^{\infty} \frac{1}{x^p}\,dx$ convergent.

For $p = 1$, we saw that $\int_1^{\infty} \frac{1}{x}\,dx$ diverges.

So, we can assume $p \neq 1$

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$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\left(\int_1^t \frac{1}{x^p}\,dx\right)$$

$$\int \frac{1}{x^p}\,dx = \int x^{-p}\,dx = \frac{x^{-p+1}}{-p+1} \quad (p \neq 1)$$

$$= \lim_{t\to\infty}\left(\frac{x^{-p+1}}{-p+1}\Bigg|_1^t\right)$$

$$= \lim_{t\to\infty}\left(\frac{t^{\boxed{-p+1}}}{-p+1} - \frac{1}{-p+1}\right)$$

For this limit to converge: $-p+1 < 0 \rightarrow \boxed{p > 1}$

diverge: $-p+1 > 0 \rightarrow \boxed{p < 1}$

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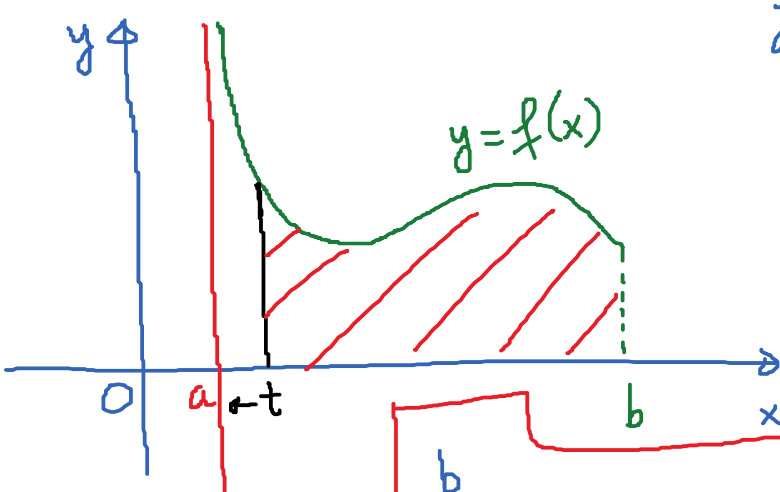
Conclusion:

$$\int_1^\infty \frac{dx}{x^p} \begin{cases} \to \text{if } p \le 1, \text{ the integral diverges} \\ \to \text{if } p > 1, \text{ the integral converges} \end{cases}$$

Type 2 Improper Integrals: Discontinuous Integrands

Scenario 1:

$\int_a^b f(x)\,dx$, f is discontinuous at the left endpoint $x=a$



$$\boxed{\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx}$$

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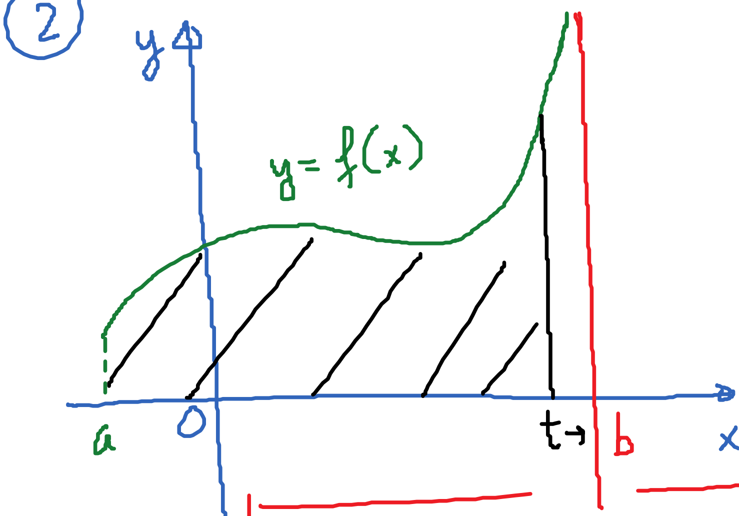
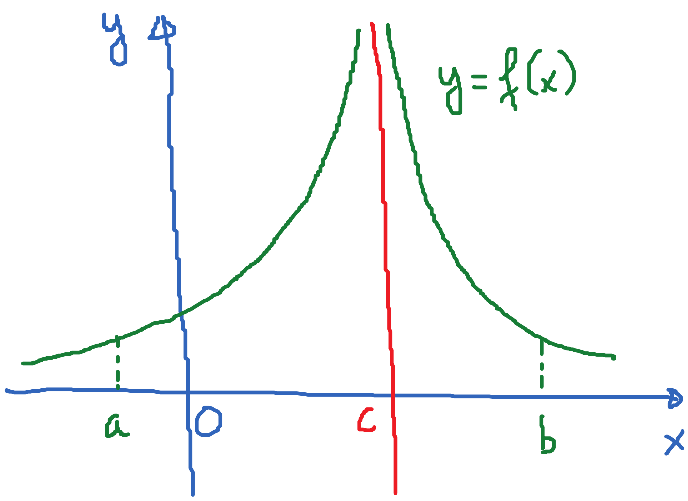
(2)



$\int_a^b f(x)\,dx$, f has a discontinuity at the right endpoint

$$\boxed{\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx}$$

(3)



$\int_a^b f(x)\,dx$, f is discontinuous at a point c in (a, b)

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \lim_{t \to c^-} \int_a^t f(x)\,dx + \lim_{t \to c^+} \int_t^b f(x)\,dx$$

Eg. Evaluate $\displaystyle\int_0^3 \frac{dx}{x-1}$.

$$\cancel{= \ln|x-1| \Big|_0^3 = \ln(2).}$$ (WRONG!)

The integrand $\dfrac{1}{x-1}$ is discontinuous at $x=1$.

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

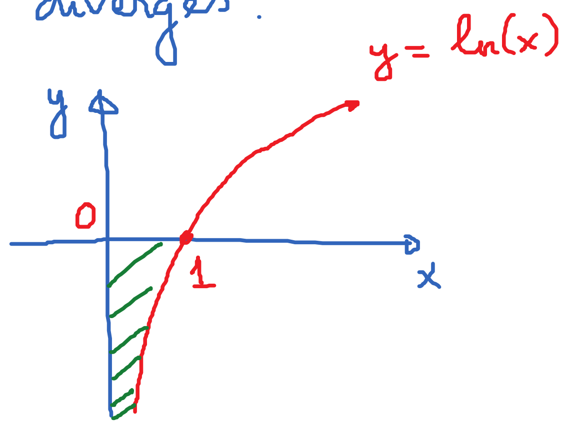
$$= \lim_{t\to 1^-}\left(\int_0^t \frac{dx}{x-1}\right) + \lim_{t\to 1^+}\left(\int_t^3 \frac{dx}{x-1}\right)$$

$$= \lim_{t\to 1^-}\left(\ln|x-1|\right)\Big|_0^t + \lim_{t\to 1^+}\left(\ln|x-1|\right)\Big|_t^3$$

$$= \underbrace{\lim_{t \to 1^-} \ln|t-1|}_{-\infty} + \lim_{t \to 1^+} \left(\ln 2 - \ln|t-1| \right)$$

Conclusion. The integral diverges.

E.g Find $\displaystyle\int_0^1 \ln x \, dx$



$$= \lim_{t \to 0^+} \int_t^1 \ln x \, dx$$

$$\begin{cases} u = \ln x \\ dv = dx \end{cases} \rightarrow \begin{cases} du = \frac{1}{x} dx \\ v = x \end{cases}$$

$$= \lim_{t \to 0^+} \left(x \ln x \Big|_t^1 - \int_t^1 dx \right)$$

$$= \lim_{t \to 0^+} \left(-\underbrace{t \ln(t)} - (1 - t) \right)$$

$$= \lim_{t \to 0^+} \left(- \frac{\ln(t)}{\frac{1}{t}} - 1 \right) = \lim_{t \to 0^+} \left(- \frac{\frac{1}{t}}{-\frac{1}{t^2}} - 1 \right)$$

L'Hopital

$$= \lim_{t \to 0^+} (t - 1) = \boxed{-1}$$