Ex. Find $\lim_{n\to\infty} a_n$ for the given sequence

(a) $a_n = \dfrac{3n^2+n+4}{5n^2-3}$; $n \geq 1$

(b) $a_n = \sqrt{n}\,\boxed{\ln\left(1+\frac{1}{n}\right)}$; $n \geq 1$ $\rightarrow 0$

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(c) $a_n = n \cdot \sin\left(\frac{1}{n}\right)$; $n \geq 1$.

(a) $\lim_{n\to\infty} \dfrac{3n^2+n+4}{5n^2-3} = \lim_{n\to\infty} \dfrac{6n+1}{10n} = \lim_{n\to\infty} \dfrac{6}{10} = \dfrac{3}{5}$.

L'Hopital

(b) $\lim_{n\to\infty} \dfrac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{\sqrt{n}}}$ $\left(\frac{\infty}{\infty}\text{ form}\right) \rightarrow$ L'Hopital

$\left(n^{-1/2}\right)' = -\frac{1}{2}n^{-\frac{3}{2}} = -\dfrac{1}{2n^{3/2}}$

$$\lim_{n\to\infty} \frac{\dfrac{1}{1+\frac{1}{n}}\cdot\left(-\frac{1}{n^2}\right)}{-\dfrac{1}{2n^{3/2}}} = \lim_{n\to\infty} \frac{2n^{3/2}}{n^2+n} = 0$$

(c) $$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\cos\left(\frac{1}{n}\right)\cdot\left(\cancel{-\frac{1}{n^2}}\right)}{\cancel{-\frac{1}{n^2}}}$$

L'Hopital

$$= \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos(0) = 1.$$

E.g. Consider the sequence $\{R^n\}_{n=1}^{\infty}$ where R is a fixed real #.

Q: For what values of R does the sequence converge?

$$\boxed{\lim_{n\to\infty} R^n = \begin{cases} 0 & \text{if } -1<R<1 \\ 1 & \text{if } R=1 \end{cases}}$$

$\{R^n\}$ converges when R is in $(-1, 1]$

Def: A sequence $\{a_n\}_{n=1}^{\infty}$ is called monotonic if it is either an increasing sequence $(a_n \leq a_{n+1})$ or a decreasing sequence $(a_n \geq a_{n+1})$

E.g. $a_n = n \, ; \, n \geq 1.$

$\{1, 2, 3, 4, 5, \dots\} \rightarrow$ increasing, hence, monotonic

$a_n = \frac{1}{n} \, ; \, n \geq 1$

$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\} \rightarrow$ decreasing, hence, monotonic

$a_n = \frac{(-1)^n}{n}.$

$\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \dots\} \rightarrow$ NOT monotonic.

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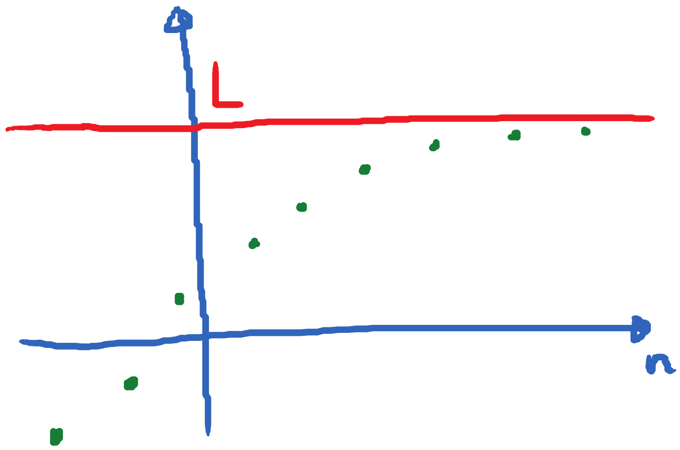
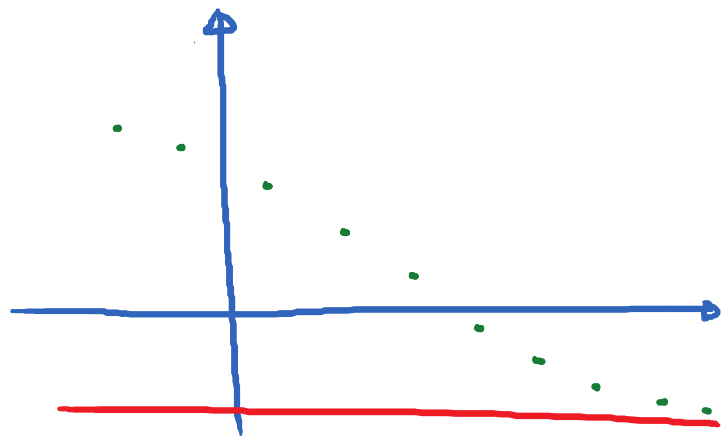
Theorem: Every bounded, monotonic sequence will converge.





Note: If $a_n = f(n)$, $f'(n) > 0 \rightarrow a_n$ increasing

$f'(n) < 0 \rightarrow a_n$ decreasing.

E.g. $a_n = \dfrac{n}{n^2+1}$; $n \geq 1$.

Q1: Is $\{a_n\}$ monotonic? Why?

Q2: Is $\{a_n\}$ bounded?

Q3: Find limit.

Q1: $f'(n) = \dfrac{n^2+1 - n\cdot(2n)}{(n^2+1)^2} = \dfrac{1-n^2}{(n^2+1)^2} \leq 0$

$\rightarrow \{a_n\}$ decreasing hence monotonic.

Q2: $\{a_n\}$ is clearly bounded below by 0.

Q3: $\lim\limits_{n\to\infty} a_n = 0$.

E.g. $\{\underbrace{\sqrt{2}}_{a_1}, \underbrace{\sqrt{2\sqrt{2}}}_{a_2}, \underbrace{\sqrt{2\sqrt{2\sqrt{2}}}}_{a_3}, \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}, \dots\}$

Q1: Is this sequence monotonic? Yes. (increasing)

Q2: Is this sequence bounded?

Q3: Find the limit?

$a_2 = \sqrt{2a_1}$; $a_3 = \sqrt{2a_2}$; $a_4 = \sqrt{2a_3}$;

$a_n = \sqrt{2a_{n-1}}$

$a_1 < 2 \rightarrow a_2 = \sqrt{2a_1} < \sqrt{2 \cdot 2} = 2$

$\rightarrow a_3 = \sqrt{2a_2} < \sqrt{2.2} = 2 \rightarrow \cdots$

Bounded above by 2.

Q3: $a_n = \sqrt{2a_{n-1}}$

$\lim_{n\to\infty} a_n = \sqrt{2 \lim_{n\to\infty} a_{n-1}}$

$L = \sqrt{2L} \longrightarrow L^2 = 2L \rightarrow L = 2$